1 Alternate State-Space Representations

As ststed earlier, there are an infinite number of possible state space realizations (models) for any given system. Some of these representations (canonical forms) are more useful than others, they are:

1. Controllable Canonical Form
2. Observable Canonical Form
3. Diagonal Canonical Form
4. Jordan Canonical Form

Given any particular representation, it is possible to transform to any of the other representations. These forms have various advantages, that simplify conversion to transfer function models and allow us to "see" the properties of controllability and observability (topics that have been covered in the last 2 lectures).

Consider the system defined by

$$\overset{(n)}{y} + a_1 \overset{(n-1)}{y} + \ldots + a_{n-1}\dot{y} + a_n y = b_0 \overset{(n)}{u} + b_1 \overset{(n-1)}{y} + \ldots + b_{n-1}\dot{u} + b_n u$$

where u is the input and y is the output. Taking the Laplace transform of both sides we get:

$$Y(s)(s^n + a_1 s^{n-1} + \ldots + a_{n-1}s + a_n) = U(s)(b_0 s^n + b_1 s^{n-1} + \ldots + b_{n-1}s + b_n)$$

whcih yields the transfer function:

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \ldots + b_{n-1}s + b_n}{s^n + a_1 s^{n-1} + \ldots + a_{n-1}s + a_n} \tag{1}$$

We will now consider the various canonical state space forms for a system with the transfer function of equantion 1.

1.1 Controllable Canonical Form

The controllable canonical form has the following layout:

$$
\begin{aligned}
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} &= \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u \\
y &= \begin{bmatrix} b_n - a_n b_0 & b_{n-1} - a_{n-1} b_0 & \cdots & b_1 - a_1 b_0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + b_0 u
\end{aligned}
$$

Controllable canonical form is useful for pole placement controller design techniques (this will be shown later).

Example 1 *Consider the system given by*

$$\frac{Y(s)}{U(s)} = \frac{s+3}{s^2+3s+2}$$

Obtain a state space representation in controllable canonical form.

Controllable Canonical, by inspection, $a_1 = 3, a_2 = 2$ and $b_0 = 0, b_1 = 1, b_2 = 3$:

$$
\begin{aligned}
\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\
y(t) &= \begin{bmatrix} 3 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}
\end{aligned}
$$

Remark 2 *Matlab (or Octave) uses the controllable canonical form by default when converting from a transfer function representation.*

```
>>num = [1 3];
>>den = [1 3 2];
>>sys = tf2ss(num,den)     %create an LTI SS object from transfer function
```

```
>>a =

     0     1
    -2    -3

b =

     0
     1

c =

     3     1

d = 0
```

1.2 Observable Canonical Form

The observable canonical form has the following layout:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_n \\ 1 & 0 & \cdots & 0 & -a_{n-1} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_n - a_n b_0 \\ b_{n-1} - a_{n-1} b_0 \\ \vdots \\ b_1 - a_1 b_0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + b_0 u$$

Note that the A matrix for observable canonical form is the transpose of the A matrix for controllable canonical form.

Example 3 *Consider the system given of example 1. Obtain state space representation in observable canonical form.*

Observable Canonical form, by inspection, $a_1 = 3, a_2 = 2$ and $b_0 = 0, b_1 =$

$1; b_2 = 3$:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Remark 4 *Matlab provides conversion to canonical observable (i.e. companion) form with the command canon.*

```
>>csys = canon(sys,'companion')
>>a =

      0      -2
      1      -3

b =

     3
     1

c =

     0      1

d = 0
```

1.3 Diagonal Canonical Form

Diagonal canonical form can be written providing the denominator polynomila of equation 1 can be factored into distinct roots. In this case, we have:

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \ldots + b_{n-1}s + b_n}{(s+p_1)(s+p_2)\cdots(s+p_n)} +$$

expanding using partial fractions gives

$$= b_0 + \frac{c_1}{s+p_1} + \frac{c_2}{s+p_2} + \cdots + \frac{c_n}{s+p_n}$$

Then the diagonal canonical form is given by

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \vdots\\ \dot{x}_n\end{bmatrix} = \begin{bmatrix}-p_1 & & & 0\\ & -p_2 & & \\ & & \ddots & \\ 0 & & & -p_n\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ \vdots\\ x_n\end{bmatrix} + \begin{bmatrix}1\\ 1\\ \vdots\\ 1\end{bmatrix}u$$

$$y = \begin{bmatrix}c_1 & c_2 & \cdots & c_n\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ \vdots\\ x_n\end{bmatrix} + b_0 u$$

Example 5 *Convert the transfer function of example 1 to diagonal canonical form:*

$$\frac{Y(s)}{U(s)} = \frac{s+3}{s^2+3s+2} = \frac{s+3}{(s+1)(s+2)}$$

so $p_1 = 1$ *and* $p_2 = 2$*,*

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} = \begin{bmatrix}-1 & 0\\ 0 & -2\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + \begin{bmatrix}1\\ 1\end{bmatrix}u(t)$$

$$y(t) = \begin{bmatrix}2 & 1\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}$$

Remark 6 *Matlab can convert to diagonal canonical (i.e. modal) form as well:*

```
>>csys = canon(sys,'modal')
>>a =

     -1        0
      0       -2


b =

     1
     1


c =
```

```
     2     -1

d = 0
```

1.4 Jordan Canonical Form

Consider the case where the polynomial equation of the denominator of equation 1 involves multiple roots. The diagonal canonical form can be rewritten as a modified diagonal form called the Jordan form. Say for example, that the we have a denominator with three identical roots, $p_1 = p_2 = p_3$, then the factored form of the transfre function is

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \ldots + b_{n-1}s + b_n}{(s+p_1)^3(s+p_4)(s+p_5)\cdots(s+p_n)}$$

partial fraction expansion yields

$$= b_0 + \frac{c_1}{(s+p_1)^3} + \frac{c_2}{(s+p_1)^2} + \frac{c_3}{(s+p_1)} + \frac{c_4}{s+p_4} + \frac{c_5}{s+p_5} + \cdots + \frac{c_n}{s+p_n}$$

the state space form is then

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \left[\begin{array}{ccc|ccc} -p1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & -p1 & 1 & \vdots & & \vdots \\ 0 & 0 & -p1 & 0 & \cdots & 0 \\ \hline 0 & \cdots & 0 & -p4 & & 0 \\ \vdots & & \vdots & & \ddots & \\ 0 & \cdots & 0 & 0 & & -pn \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + b_0 u$$