Engi 9867 Advanced Computing Concepts for Engineering

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Total marks: 107
Name:
Student #:

Closed book. No calculators or other electronic aids are permitted, other than simple time keeping devices. Paper, English only dictionaries are allowed, as are paper English to .+ dictionaries.

If you need extra space for answers or for any other reason, ask for a yellow booklet. All yellow booklets and exam papers will be collected.

- A *k-colouring* of an undirected graph $G = (V, E)$ is a total function f from V to a set of size $k \in \mathbb{N}$, such that, for all $\{v, w\} \in E$, $f(v) \neq f(w)$.
- *Graph Colouring* is the following NP-Complete decision problem:
  - Input: An undirected graph $G = (V, E)$ and an integer k.
  - Size: The size of the graph.
  - Question: Is there is a k-colouring of G.
- A *clique* is a set of vertices in an undirected graph, each of which is connected to all the others by an edge.
- The *Clique Problem* is the following NP-Complete problem: Given a graph G and a number k, does G contain a clique of size k?
- A *Hamiltonian cycle* in $G = (V, E)$ is a path in that starts and ends at the the same vertex, but otherwise contains each vertex exactly once.
- The *Hamiltonian Cycle Problem* is the NP-Complete problem: Given a graph G, does it have a Hamiltonian cycle.
- A propositional formula is in *CNF* iff it is a conjunction of disjunctions and each disjunct is either a variable or the negation of a variable.
- A propositional formula is in *3CNF* iff it is in CNF and, furthermore, each conjunct contains three disjuncts.
- A propositional formula is *satisfiable* iff there is a way to assign boolean values to its variables so that the formula evaluates to **true**.
- *SAT* is the following NP-Complete problem: Given a propositional formula ϕ in CNF form, is ϕ satisfiable.
- *3SAT* is the following NP-Complete problem: Given a propositional formula ϕ in 3CNF form, is ϕ satisfiable.

Student #: ____________________

Q0 [10] The prefix-closure of a language M is the set of all strings that start a string in M. I.e. $\mathrm{pref}(M) = \{s \mid (\exists t \cdot s; t \in M)\}$. Prove that if M is a regular language, so is $\mathrm{pref}(M)$.

Q1 [20]
Consider the following specification

$$f = \left\langle x' = \mathrm{fib}(n) \right\rangle$$

with x and n being natural number variables. $\mathrm{fib}(i)$ is defined by

$$\mathrm{fib}(0) = \mathrm{fib}(1) = 1$$
$$\mathrm{fib}(i+2) = \mathrm{fib}(i) + \mathrm{fib}(i+1), \text{ for all } i \in \mathbb{N}$$

Read the whole question before beginning to answer

(a) [5] Propose an invariant $\mathcal{I}$
$\mathcal{I}$ is

(b) [5] Give a command that refines $m = \langle \mathcal{I}' \wedge n' = n \rangle$
$m \sqsubseteq$

(c) [5] Give an expression $\mathcal{A}$ and such that $\mathcal{I} \wedge \neg\mathcal{A} \Rightarrow x = \mathrm{fib}(n)$ is valid
$\mathcal{A}$ is

(d) [5] Give a command that refines $h = \langle \mathcal{A} \wedge \mathcal{I} \Rightarrow \mathcal{I}' \wedge n' = n \rangle$ and so that the loop terminates
$h \sqsubseteq$

Q2 **[18]** A $G' = (V', E')$ is a *subgraph* of a graph $G = (V, E)$ if and only if $V' \subseteq V$ and $E' \subseteq E$ and G' is a graph. Two graphs $G' = (V', E')$ and $H = (W, F)$ are *isomorphic* if $|V'| = |W|$ and $|E'| = |F|$ and there is an invertible total function[1] f from V' to W such that, for all $u, v \in V'$, $\{u, v\} \in E'$ if and only if $\{f(u), f(v)\} \in F$.

Show the following problem is NP-Complete: Given graphs G and H, is there a subgraph of G that is isomorphic to H.

(a) **[4]** Give a short, convincing argument that the problem is in **NP**.

(b) **[2]** Pick an NP-Complete problem to reduce from. (See page 0 for some suggestions.)

(c) **[6]** What are the properties of the required transformation. (What are its inputs, what are its outputs, what is the required relation between its inputs and outputs, what time limitation is there on the transformation?)

(d) **[6]** Define a transformation that will meet these requirements. (I.e. define each output of the transformation in terms of its inputs.)

[1] A total function $f \in S \to T$ is *invertible* if there is a total function $g \in T \to S$ so that, for all s,$g(f(s)) = s$, and, for all t, $f(g(t)) = t$.

Q3 [12]

Consider the following language L of sequences over the set $\{0, 1, 2, 3, a\}$. Any finite sequence that obeys the following rules is in the language

- Every sequence ends with 0 (therefore no sequence in L is empty).
- 0 only appears as the last symbol in a sequence.
- Immediately before each 1 there is one and only one a.
- Immediately before each 2 there are two and only two as.
- Immediately before each 3 there are three and only three as.

Here are some examples and counter-examples

$$0\ ,\ aa20\ ,\ a1aaa3aa20 \in L$$
$$\epsilon\ ,\ aa2\ ,\ a1a20\ ,\ a1a20\ ,\ aaaa30\ a1a10a10 \notin L$$

(a) [6] Give a regular expression for this language.

(b) [6] Draw a DFR for this language. Be sure to indicate which states are final.

Q4 [9]

Suppose N is a constant of type $\mathbb{N}$ and that you have the following signature

$$\Sigma = \{\text{“}x\text{”} \mapsto \mathbb{Z}, \text{“}a\text{”} \mapsto (\{0, ..N\} \to \mathbb{Z})\,, \text{“}b\text{”} \mapsto (\{0, ..N\} \to \mathbb{Z})\}$$

Express the following informal specifications as formal specifications

(a) [3] Set x to some item of a. Don't change a or b.

(b) [3] Copy each item of array a to array b.

(c) [3] Assuming there is one, set x to the smallest index where a and b have the same value, if there is no index where a and b have the same value, it doesn't matter what happens.

Q5 [20]
Let G_0 be the context free grammar $(\{S', D, T, F\}, \{\mathsf{var}, \mathsf{id}, \mathsf{n}, :, [,], *, (,), \$\}, P, S')$, where P is

$$\begin{aligned} S' &\to D\ \$ \\ D &\to \mathsf{var}\ \mathsf{id} : T \\ T &\to T\ [\ \mathsf{n}\ ] \\ T &\to F \\ F &\to \mathsf{id} \\ F &\to *\ F \\ F &\to (\ T\ ) \end{aligned}$$

(a) [4] Draw a derivation tree for the string $\mathsf{var}\ \mathsf{id} : *\ \mathsf{id}\ [\ \mathsf{n}\ ]$

(b) [4] Give a derivation for the same string

(c) [7] For each production, what is the selector set?

Production	Selector set
$S' \to D\ \$$	{ }
$D \to \mathsf{var}\ \mathsf{id} : T$	{ }
$T \to T\ [\ \mathsf{n}\ ]$	{ }
$T \to F$	{ }
$F \to \mathsf{id}$	{ }
$F \to *\ F$	{ }
$F \to (\ T\ )$	{ }

(d) [5] G_0 is not LL(1). Give a grammar that is equivalent to G_0 (i.e. describes the same language), but that is $LL(1)$.

Q6 [18]
(a) [3] What is the exact meaning of $f \in O(g)$?

(b) [3] What is the exact meaning of $f \in \Omega(g)$?

(c) [3] Define the set **NP**

(d) [3] Suppose that problems P and Q are in **NP**. Define what it means for P to be polynomially reducible to Q.

(e) [3] Suppose a problem Q is in **NP**. Define what it means for Q to be **NP**-complete?

(f) [3] Suppose Q is in **NP**, P is **NP**-complete and that P is reducible to Q. From the definitions above, prove that Q is also **NP**-complete.