Application: AVL trees and the golden ratio

AVL trees are used for storing information in an efficient manner.

- We will see exactly how in the data structures course.
- This slide set takes a look at how high an AVL tree of a given size can be.

The golden ratio

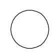
The golden ratio is an irrational number $\phi = \frac{1+\sqrt{5}}{2} \cong 1.618$ with many interesting properties. Among them

- $\phi - 1 = 1/\phi$
- $\phi = 1 + \frac{1}{1+\frac{1}{1+\frac{1}{\ddots_1}}}$
- ϕ turns up in many geometric figures including pentagrams and dodecahedra
- It is the ratio, in the limit, of successive members of the Fibonacci sequence

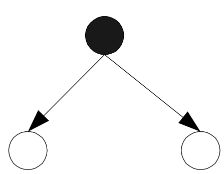
Binary trees

A binary tree is either

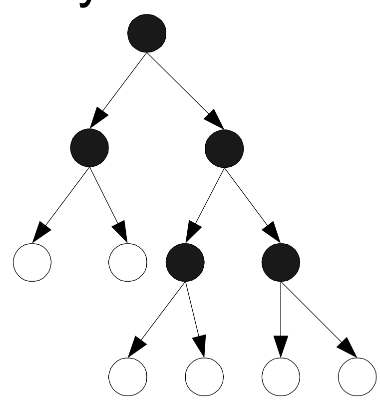
- The empty binary tree, for which I'll write ◯
- Or a point (called a **node**) connected to two smaller binary trees (called its **children**)
- The children must not share any nodes.



An empty binary tree



A nonempty binary tree

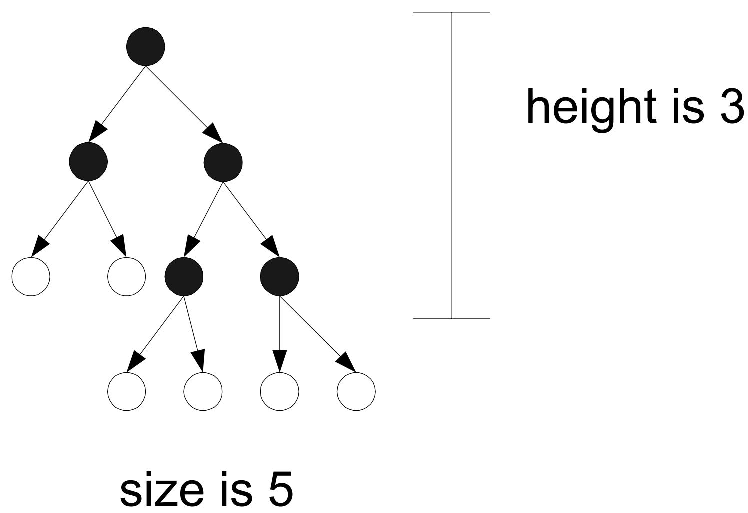


Another nonempty binary tree

The height and size of a binary tree

The **size** of a binary tree is the number of nodes it has.

The **height** of a binary tree is number of levels of nodes it has



Note that ◯ has height 0 and size 0.

Clearly a binary tree of size n can have a height of up to n.

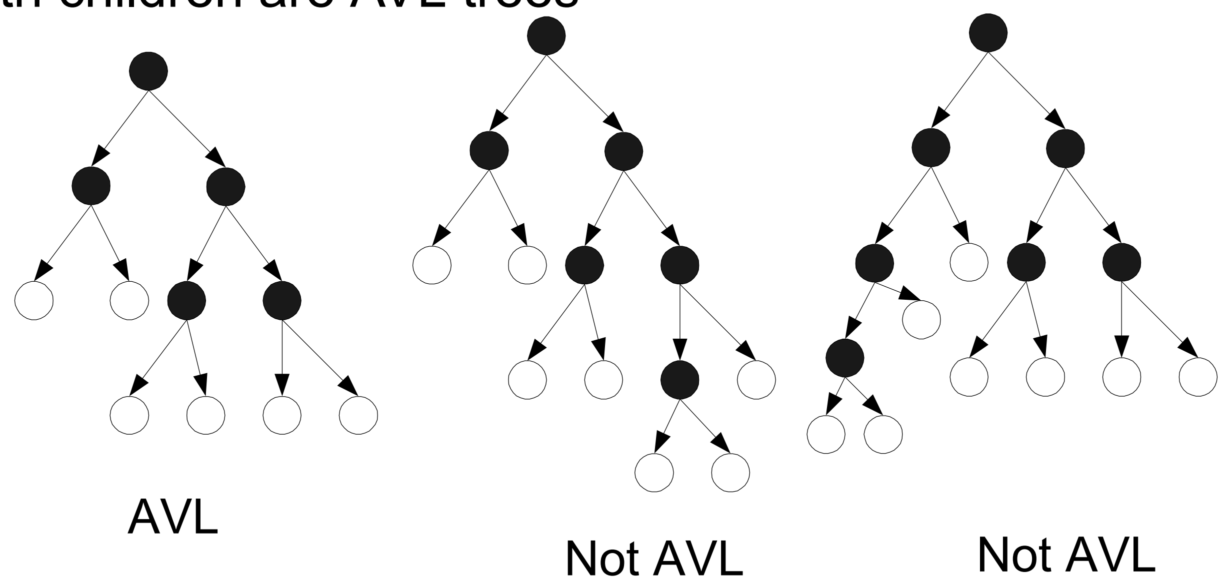
When binary trees are used to store data:

- The amount of information stored is proportional to size of tree
- The time to access data is proportional to the height

AVL trees

AVL trees are binary trees with the following restrictions.

- The empty tree is an AVL tree
- A nonempty binary tree is AVL if
  - $*$ the height difference of the children is at most 1, and
  - $*$ both children are AVL trees



The question

We wish to access large amounts of data quickly.

- Remember amount of information is proportional to size of tree
- and access time is proportional to the height of the tree.

So the question is how high can an AVL tree of a given size be?

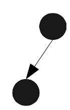
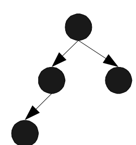
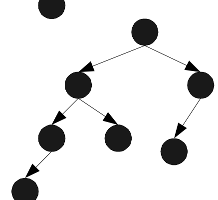
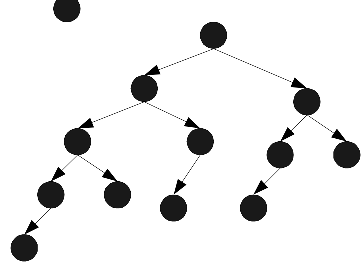
We start by asking a closely related question:

- How small can an AVL tree of a given height be?

How small can an AVL tree of a given height be?

Let’s make a table with the smallest AVL tree of each height

(empty trees are implied)

Height	Smallest tree	Size
0		0
1		1
2		2
3		4
4		7
5		12

The $\mathrm{minsize}$ function

In the table, each tree (of height $h > 1$) has, as children, smallest trees of heights $h-2$ and $h-1$

So we have

$$\begin{aligned} \mathrm{minsize}(0) &= 0 \\ \mathrm{minsize}(1) &= 1 \\ \mathrm{minsize}(h) &= \mathrm{minsize}(h-1) + \mathrm{minsize}(h-2) + 1, \text{ for } h \geq 2 \end{aligned}$$

Note the recurrence is not homogeneous.

Try a few values

$$0, 1, 2, 4, 7, 12, 20, 33, 54$$

Compare with the Fibonacci sequence

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55$$

We find

$$\mathrm{minsize}(h) = \mathrm{fib}(h+1) - 1$$

where

$$\begin{aligned} \mathrm{fib}(0) &= 1 \\ \mathrm{fib}(1) &= 1 \\ \mathrm{fib}(n) &= \mathrm{fib}(n-1) + \mathrm{fib}(n-2), \text{ for } n \geq 2 \end{aligned}$$

We can prove this by (complete induction).

Since fib is defined by a linear homogeneous recurrence relation of degree 2 we can solve it

$$\mathrm{fib}(n) = \frac{1}{\sqrt{5}} \times \phi^{n+1} - \frac{1}{\sqrt{5}} \times (\frac{-1}{\phi})^{n+1} \quad \text{for all } n \in \mathbb{N}$$

where

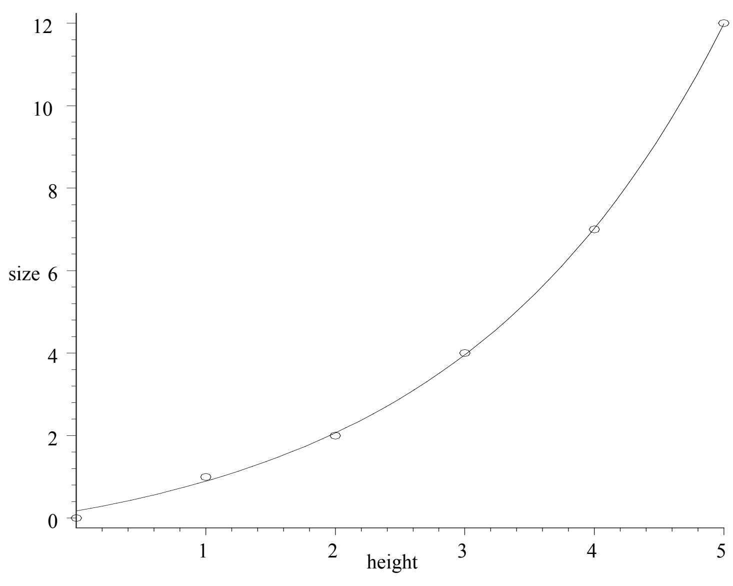
$$\phi = \frac{1+\sqrt{5}}{2}$$

Consider $\frac{1}{\sqrt{5}} \times \phi^{n+1} - \frac{1}{\sqrt{5}} \times (\frac{-1}{\phi})^{n+1}$ for $n \in \mathbb{R}$ and $n \geq 0$.

The first term is real, the second is complex.

As n gets big, the complex term becomes small.

So we get $\mathrm{minsize}(h) \cong \frac{1}{\sqrt{5}} \times \phi^{h+2} - 1$



$\mathrm{minsize}(h)$ dots $\qquad \frac{1}{\sqrt{5}} \times \phi^{h+2} - 1$ line

The maximum height per given size

Height	0	1	2	3	4	5
Min size	0	1	2	4	7	12

Let h' be the height of a tree of size s'. We know that for all h,

$$h' \geq h \rightarrow s' \geq \text{minsize}(h)$$

Contrapositively: For all h,

$$s' < \text{minsize}(h) \rightarrow h' < h$$

Size	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Max height	0	1	2	2	3	3	3	4	4	4	4	4	5	5	5

Note that for s such that $\text{minsize}(h-1) < s \leq \text{minsize}(h)$

$$\text{maxheight}(s) = h$$

$\text{maxheight}(s)$ is approximately an inverse of $\text{minsize}(h)$

So invert $\frac{1}{\sqrt{5}} \times \phi^{h+2} - 1$

$$
\begin{aligned}
& s = \frac{1}{\sqrt{5}} \times \phi^{h+2} - 1 \\
\Leftrightarrow\ & \sqrt{5}\,(s+1) = \phi^{h+2} \\
\Leftrightarrow\ & \log_\phi \sqrt{5}\,(s+1) = h+2 \\
\Leftrightarrow\ & \log_\phi \sqrt{5}\,(s+1) - 2 = h \\
\Leftrightarrow\ & \log_\phi 2 \times \log_2(s+1) + \log_\phi \sqrt{5} - 2 = h
\end{aligned}
$$

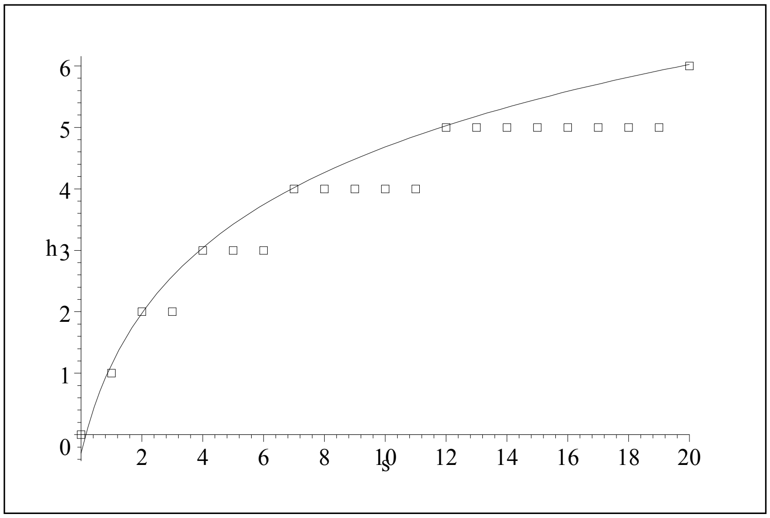
so $\mathrm{maxheight}(s) \cong 1.44 \times \log_2(s+1) - 0.3$

For example

$$
\begin{aligned}
\mathrm{maxheight}(10^6) &\cong 29 \\
\mathrm{maxheight}(10^9) &\cong 43 \\
\mathrm{maxheight}(10^{12}) &\cong 58
\end{aligned}
$$

This means large amounts of data can be accessed in a small amount of time, if we store the data in AVL trees.

Graphing maxheight



$\mathrm{maxheight}(s)$ dots $\quad 1.44 \times \log_2(s+1) - 0.3$ line