Sets

Reading: 2.1, 2.5 (optional). 2.9.

Informal defn: A **set** is a mathematical entity that is a collection of 0 or more mathematical entities.

Note: Formal definitions of the set concept are rather complex in order to avoid certain problems such as "Russell's Paradox" which are beyond the scope of the course. We will use the above informal definition.

Defn: The entities that are in a set are called its **elements**

Notn: We can write a set as a list of its elements, using "curly braces" to surround and commas to separate.

Examples:

- $\{2, 3, 5, 7\}$ is the set whose elements are the first 4 primes.
- $\{F, T\}$ is the set whose elements are the two boolean values
- $\{42\}$ is the set whose only element is the number 42.

Notn: We will write $x \in S$ to mean: x is an **element of** set S and $x \notin S$ to mean $\neg(x \in S)$.

Examples:

- $5 \in \{2, 3, 5, 7\}$
- $11 \notin \{2, 3, 5, 7\}$

Defn: Two sets are **equal** if and only if they have the same elements. We will use the notation $S = T$.

Examples:

- $\{2, 3, 5, 7\}$ is the set whose elements are the first 4 primes.
- $\{7, 3, 2, 5\}$ is the same set.
- $\{2, 2, 3, 3, 5, 5, 7\}$ is also the same set.
- So $\{2, 3, 5, 7\} = \{7, 3, 2, 5\} = \{2, 2, 3, 3, 5, 5, 7\}$

Note that $\{2\} \neq 2$. Likewise $\{\{2\}\} \neq \{2\}$

Some familiar sets

- $\mathbb{N}$ The set of all natural numbers (i.e. of all nonnegative integers).
- Note $0 \in \mathbb{N}$
- $\mathbb{Z}$ The set of all integers
- $\mathbb{Q}$ The set of all rational numbers
  - $*$ I.e. real numbers that can be expressed by dividing

an integers by a positive integer

- $\mathbb{R}$ The set of all real numbers
- $\mathbb{C}$ The set of all complex numbers

We assume that there is a **universal set** U that contains all entities that might be in any set of interest. When we use the universal set, we should be careful to define what it is. For example, if we are interested only in sets of integers, we should define that $U = \mathbb{Z}$

Set builder notation: Suppose that x is a variable and P is a proposition that *depends* on x, we write:

$$\{x \mid P\}$$

to mean that set of all elements u of U such that P evaluates to T when the value u is substituted for the variable x in the expression P.

Examples (assuming $U = \mathbb{C}$):

- What is $\{x \mid x^2 = 4\}$? $\boxed{}$
- What is $\{x \mid x^2 = 4 \land x \in \mathbb{N}\}$? $\boxed{}$
- What is $\{y \mid y = \sum_{i=1}^{N} i$, for some $N \in \{1, 2, 3, 4, 5\}\}$?
  - $\boxed{}$
- What is $\{x \mid x \in \mathbb{C} \land |x| \leq 1\}$?

- What is $\{x \mid x \in \mathbb{C} \wedge |x| = 1\}$?

Special notation for integers

- Example: $\{4, 5, ..., 10\} = \{4, 5, 6, 7, 8, 9, 10\}$
- Example: $\{1, 3, ..., 9\} = \{1, 3, 5, 7, 9\}$

Special notation for sets of real numbers

- $[x, z] = \{y \mid y \in \mathbb{R} \wedge x \leq y \leq z\}$
- $(x, z) = \{y \mid y \in \mathbb{R} \wedge x < y < z\}$

The **empty set**. There is one set that has no elements. We write it as $\{\}$ or as $\emptyset$.

Check:

- What is $\{x \mid F\}$?
- What is $\{x \mid T\}$?

Is $\{\emptyset\} = \emptyset$? No. $\{\emptyset\}$ has one element whereas $\emptyset$ does not , so they don't have the same number of elements.

Subset: Set A is a **subset** of set B if and only if every element of A is also an element of B. We write $A \subseteq B$.

Examples

- $\{2\} \subseteq \{-2, 2\}$
- $\{-2\} \subseteq \{-2, 2\}$

- $\{-2, 2\} \subseteq \{-2, 2\}$
- $\emptyset \subseteq \{-2, 2\}$
- In fact $\emptyset \subseteq A$ for any set A
- Are the any other subsets of $\{-2, 2\}$? $\boxed{}$
- $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$

Note

- $\emptyset \subseteq A$ for all sets A
- $A \subseteq A$, for all sets A
- $A \subseteq B \wedge B \subseteq A \Leftrightarrow A = B$

Proper subset: Set A is a **proper subset** of set B if and only if $A \subseteq B$ but $A \neq B$. We write $A \subset B$.

Examples

- $\{2\} \subset \{-2, 2\}$
- $\{-2\} \subset \{-2, 2\}$
- $\emptyset \subset \{-2, 2\}$
- In fact $\emptyset \subset A$ for any set A other than $\emptyset$
- $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$

Union: The **union** of sets A and B (written $A \cup B$) contains any element contained either by A or by B.

$$A \cup B = \{x \mid x \in A \vee x \in B\}$$

Intersection: The **intersection** of sets A and B (written $A \cap B$) contains any element contained both by A and by B

$$A \cap B = \{x \mid x \in A \wedge x \in B\}$$

Examples:

- What is $\{-2, 2\} \cup \{2, 3, 5, 7\}$?
- What is $\{-2, 2\} \cap \{2, 3, 5, 7\}$?
- What is $\{x \mid x \in \mathbb{C} \wedge |x| = 1\} \cap \mathbb{R}$?
- What is $[x_0, z_0] \cup [x_1, z_1]$?
- What is $[x_0, z_0] \cap [x_1, z_1]$?

Disjoint: Sets A and B are said to be **disjoint** iff their intersection is null

$$A \cap B = \emptyset$$

Difference: The **difference** of sets A and B (written $A - B$) contains all elements of A not in B.

Complement: The complement of set A (written $\overline{A}$) contains all elements of the universal set, U, not in A . I.e. $\overline{A} = U - A$.

Examples

- Example: $\mathbb{N} - \{0, 2, 4, ...\}$ is what?
- If $U = \mathbb{C}$ then what is $\overline{\{x \mid |x| \leq 1\}}$
- What is $A - B$ if A and B are disjoint?

Union and intersection of multiple sets

Suppose that $f(0)$, $f(1)$, $f(2)$ are sets (f is a set valued function), then we write

$$\cup_{i \in \{0,1,2\}} f(i)$$

to mean

$$f(0) \cup f(1) \cup f(2)$$

and

$$\cap_{i \in \{0,1,2\}} f(i)$$

to mean

$$f(0) \cap f(1) \cap f(2)$$

More generally, suppose that for each i in a set Υ $f(i)$ is a set (f is a function from Υ to a set of sets) then

$$\cup_{i\in\Upsilon} f(i) = \{s \mid s \in f(j), \text{ for some } j \in \Upsilon\}$$
$$\cap_{i\in\Upsilon} f(i) = \{s \mid s \in f(j), \text{ for all } j \in \Upsilon\}$$

Example: Suppose that $\Upsilon = \{0, 1, 2\}$ and $f(i) = \{n \mid n \in \mathbb{N} \wedge \text{the remainder of } n \text{ divided by 3 is } i\}$

- What is $f(0)$?
- What is $f(1)$?
- What is $f(2)$?
- What is $\cup_{i\in\Upsilon} f(i)$?
- What is $\cap_{i\in\Upsilon} f(i)$?

Suppose $g(i) = \{n \mid n \in \mathbb{N} \wedge n = k \times i, \text{ for some } k \in \{2, 3, 4, ...\}\}$, for $i \in \{2, 3, 4, ...\}$

- What is $g(2)$?
- What is $g(3)$?
- What is $\cup_{i\in\{2,3,4,...\}} g(i)$?
- What is $\{2, 3, 4, ...\} - \cup_{i\in\{2,3,4,...\}} g(i)$?

Cardinality: If a set has only a finite number of elements, then its cardinality is that number.

Examples

- $|\{2,3,5,7\}| = \square$
- $|\{2,2,2\}| = \square$
- $|\emptyset| = \square$
- $|A \cup B| \ \square \ |A| + |B|$

Infinite Sets [optional]

If there is no number in $\mathbb{N}$ that represents the number of elements in S then we say that S is an **infinite** set.

Example $\mathbb{N}$ is an infinite set. So are $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$.

We can still talk about the cardinality of an infinite set. We say that $|A| \leq |B|$ iff there is a function f from A to B and such that for any $a, b \in A$, if $a \neq b$ then $f(a) \neq f(b)$.

We say that $|A| = |B|$ iff $|A| \leq |B|$ and $|B| \leq |A|$.

Example:

- $|\mathbb{N}| = |\mathbb{Z}|$. Why?
- $|\mathbb{N}| \leq |\mathbb{R}|$. Why ?
- Challenge: show that if $|A| \leq |B|$ and $|B| \leq |C|$ then $|A| \leq |C|$, for any sets A, B, C
- Challenge: show that $|\mathbb{N}| = |\mathbb{Q}|$

- Challenge: show that: $|\mathbb{R}| = |\mathbb{C}|$
- Challenge: show $|\mathbb{N}| \neq |\mathbb{R}|$

Power Set: A set can contain other sets as elements. For example $\{\emptyset, \{1\}, 1\}$ contains two sets and a number.

The **power set** of a set A is the set of all subsets of A. We write $\mathcal{P}(A)$.

Examples

- $\mathcal{P}(\{1,2\})$ is what? $\{\emptyset, \{1\}, \{2\}, \{1,2\}\}$
- $\mathcal{P}(\{1\})$ is what?
- $\mathcal{P}(\emptyset)$ is what?
- If $|A| = n \in \mathbb{N}$ then what is $|\mathcal{P}(A)|$?

Tuples: For $n \in \mathbb{N}$, an n-tuple is a sequence of n entities.

For example, Cartesian coordinates (x, y) are 2-tuples.

A student record might consist of a name student number and a mark $(\textit{jones}, 2001314592, A^+)$. This is a 3-tuple.

Note that while $\{1,2\} = \{2,1\}$ we have $(1,2) \neq (2,1)$.

Cartesian product: Given sets A and B, $A \times B$ is the set of all 2-tuples (a, b) such that $a \in A$ and $b \in B$

$A \times B = \{x \mid x = (a,b) \wedge a \in A \wedge b \in B,$ for some a and $b\}$

Example

- $\{1,2\} \times \{a,b\} = \{(1,a),(1,b),(2,a),(2,b)\}$
- $\{0,1,..,31\} \times \{jan, feb, ..., dec\}$
  $=$

More generally, if $S_0, S_1, ..., S_{n-1}$ are n sets then $S_0 \times S_1 \times \cdots \times S_{n-1}$ is the set of all n-tuples

$$(a_0, a_1, ..., a_{n-1})$$

such that $a_i \in S_i$ for all $i \in \{0,1,...,n-1\}$

Example:

- $\{1,2\} \times \{a,b\} \times \{F,T\} =$
- $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$ is a three dimensional space.
- set-of-all-names$\times$set-of-all-student-numbers$\times$set-of-all-marks is a set of possible student records
- $\{p \mid p = (x,y) \in \mathbb{R} \times \mathbb{R} \wedge \sqrt{x^2+y^2} = 1$, for some x and $y\}$ is what?
- We could write instead $\{(x,y) \in \mathbb{R} \times \mathbb{R} \mid \sqrt{x^2+y^2} = 1\}$
- What is $\{(x,y) \in \mathbb{R} \times \mathbb{R} \mid y = \sin x\}$?

Proofs about sets

Showing a particular entity is (or is not) a member of a set

We show that the criterion for membership is (or is not) met by the entity

Example: $\mathbb{Q}$ is the set

$$\{x \mid x = p/q \wedge q \neq 0, \text{ for some } p \in \mathbb{Z}, \text{ and } q \in \mathbb{Z}\}$$

To show a specific number, say $1.\overline{234}$ is in $\mathbb{Q}$ we must find specific p and q that meet the criteria. Since

$1.\overline{234} \times 999 = 1.\overline{234} \times 1000 - 1.\overline{234} = 1234.\overline{234} - 1.\overline{234} = 1233$

Hence

$$\frac{1233}{999} = 1.\overline{234}$$

so $p = 1233$ and $q = 999$ fulfill the criterion.

$\square$

Showing $A = B$

Method 0 for showing $A = B$. Ping-pong proof

- Show that $A \subseteq B$, and then

- show that $B \subseteq A$.

Method 1 for showing $A = B$. Use IFF

Let x be an arbitrary member of the universe U.
Show that $x \in A \Leftrightarrow x \in B$

Method 2 for showing $A = B$. Use Laws.

For each equivalence law for propositional logic there is an equivalent law for sets.
Example: Consider $\overline{A \cup B}$. We have (for arbitrary x in the universe U)

$$
\begin{aligned}
& x \in \overline{A \cup B} \\
\Leftrightarrow\ & \neg(x \in A \cup B) \\
\Leftrightarrow\ & \neg(x \in A \vee x \in B) \\
\Leftrightarrow\ & \neg(x \in A) \wedge \neg(x \in B) \text{ De Morgan} \\
\Leftrightarrow\ & x \in \overline{A} \wedge x \in \overline{B} \\
\Leftrightarrow\ & x \in \overline{A} \cap \overline{B}
\end{aligned}
$$

So (by method 1) we have a De Morgan law for sets $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

□

By the same token, any equivalence law for propositions turns into an equality law for sets.

See book for extensive listing of equality laws.

Showing $A \subseteq B$

Method 0 of showing $A \subseteq B$

Use a variable (e.g. x) to represent an arbitrary member of A.

Show that $x \in B$.

Example:

Let A be the set of numbers that can be represented by repeating decimals of the form $0.\overline{b_0 b_1 ... b_{p-1}}$. Show $A \subseteq \mathbb{Q}$.

Let x be an arbitrary member of A. We know that for some $b \in \mathbb{N}$, and $p \in \{1, 2, 3, ...\}$

$$x = \frac{b}{10^p} + \frac{b}{10^{2p}} + ...$$

Consider multiplying x by 10^p we get

$$\begin{aligned} & 10^p \times x \\ = \ & 10^p x \\ = \ & 10^p \left(\frac{b}{10^p} + \frac{b}{10^{2p}} + \ldots \right) \\ = \ & b + \frac{b}{10^p} + \frac{b}{10^{2p}} + \ldots \\ = \ & b + x \end{aligned}$$

So

$$x = \frac{b}{(10^p - 1)}$$

Since $b \in \mathbb{Z}$ and $(10^p - 1) \in \mathbb{Z}$ and $(10^p - 1) \neq 0$ we see by the definition of $\mathbb{Q}$ that $x \in \mathbb{Q}$.

Hence $A \subseteq \mathbb{Q}$

$\square$

Method 1 of showing $A \subseteq B$

Show that $B = A \cup C$ for some C.

Example: Show that $X \cap \overline{Y} \subseteq X \cap \overline{(Y \cap Z)}$ for any sets X, Y, and Z

Rewrite the RHS

$$\begin{aligned} & X \cap \overline{(Y \cap Z)} \\ = \; & X \cap (\overline{Y} \cup \overline{Z}) \text{ De Morgan} \\ = \; & (X \cap \overline{Y}) \cup (X \cap \overline{Z}) \text{ Distribute } \cap \text{ over } \cup \end{aligned}$$

Now $X \cap \overline{Y} \subseteq (X \cap \overline{Y}) \cup (X \cap \overline{Y})$ so $X \cap \overline{Y} \subseteq X \cap \overline{(Y \cap Z)}$

$\square$

Method 2 of showing $A \subseteq B$

Show $A = B \cap C$, for some C.