Recurrence Relations

Reading: Gossett Sections 7.1 and 7.2.

Definition: A one-way infinite sequence is a function from the natural numbers to some other set.

E.g.

$$\mathit{fib}(0) = 1, \mathit{fib}(1) = 1, \mathit{fib}(2) = 2, fib(3) = 3, \mathit{fib}(4) = 5, \mathit{fib}(5) = 8, .$$
$$q(0) = 1, q(1) = 2, q(2) = 4, q(3) = 8, ...$$
$$pr(0) = 2, pr(1) = 3, pr(2) = 5, pr(3) = 7, ...$$

Definition: A recurrence relation is an equation that defines all members of a sequence past a certain point in terms of earlier members. That is an equation

$$a(n) = F, \text{ for all } n \in \{n_1, n_1 + 1, \cdots\}$$

where F is an expression combining only $a(n-1)$, $a(n-2)$, ..., $a(0)$.

Examples

- $\mathit{fib}(n) = \mathit{fib}(n-1) + \mathit{fib}(n-2)$
- $q(n) = 2 \times q(n-1)$

If we conjoin enough (n_1) base cases with the recurrence relation, then together they define a sequence.

Examples:

- $\mathit{fib}(0) = 1$
- $\mathit{fib}(1) = 1$
- $q(0) = 1$

Substitute and simplify method

Consider the sequence defined by

$$\begin{aligned} a(0) &= 5 \\ a(n) &= 2a(n-1) - 3, \text{ for } n \geq 1 \end{aligned}$$

We can reason (rather informally) that.

$$\begin{aligned} & a(n) \\ =\ & 2a(n-1) - 3 \\ =\ & 2(2a(n-2) - 3) - 3 \text{ substitution} \\ =\ & 4a(n-2) - 2 \cdot 3 - 3 \text{ simplify} \\ =\ & 4(2a(n-3) - 3) - 2 \cdot 3 - 3 \text{ substitution} \\ =\ & 8a(n-3) - 4 \cdot 3 - 2 \cdot 3 - 3 \text{ simplify} \\ & \vdots \\ =\ & 2^k a(n-k) - 3(2^{k-1} + 2^{k-2} + \cdots + 1) \\ & \vdots \\ =\ & 2^n a(0) - 3\left(2^{n-1} + 2^{n-2} + \cdots + 1\right) \\ =\ & 2^n \cdot 5 - 3 \cdot (2^n - 1) \text{ since } \sum_{i=0}^{n-1} 2^i = 2^n - 1 \\ =\ & 2 \cdot 2^n + 3 \\ =\ & 2^{n+1} + 3 \end{aligned}$$

If there is any doubt, we can prove the result by induction.

Unfortunately the substitute and simplify method does not always give an easy to simplify result.

Linear Homogeneous Recurrence Relations with Constant Coefficients of Degree k

Definition: A *linear homogeneous recurrence relation with constant coefficients* (LHRRCC) is a recurrence relation whose RHS is a sum of terms each of the form

$$c \cdot a(n-b)$$

where $c \in \mathbb{C}$ and $b \in \mathbb{N}$ are constants. □

Definition: The *degree* of a LHRRCC is the maximum b value for which the c value is not 0. I.e. if we have

$$a(n) = c_1 a(n-1) + c_2 a(n-2) + \cdots + c_k a(n-k)$$

with $c_k \neq 0$, then the degree is k.

When the degree is 2

Consider the degree 2 case. The RR is

$$a(n) = c_1 a(n-1) + c_2 a(n-2) \qquad \text{(*)}$$

Suppose there is a solution for the recurrence relation of the form

$$a(n) = \theta r^n, \text{ for all } n \in \mathbb{N}$$

for some $\theta \neq 0$ and some $r \neq 0$.

Then substituting into (*) we get (for any $n \in \mathbb{N}$)

$$\theta r^n = c_1 \theta r^{n-1} + c_2 \theta r^{n-2}$$

Thus

$$r^n - c_1 r^{n-1} - c_2 r^{n-2} = 0$$

Thus

$$r^{n-2} \cdot (r^2 - c_1 r - c_2) = 0$$

So r is a root of the polynomial

$$x^2 - c_1 x - c_2 \qquad (**)$$

Conversely, if r is root of the polynomial

$$x^2 - c_1 x - c_2 \qquad (**)$$

then for any θ and any $n \in \mathbb{N}$

$$\theta r^n = c_1 \theta r^{n-1} + c_2 \theta r^{n-2}$$

so

$$a(n) = \theta r^n$$

is a solution to (*).

The characteristic polynomial of (*) is (**).

We have proved the following theorem:

Theorem:

If r is a root of the characteristic polynomial

$$x^2 - c_1 x - c_2$$

then, for any $\theta \in \mathbb{C}$, the sequence

$$a(n) \triangleq \theta r^n, \text{ for all } n \in \mathbb{N}.$$

is a solution to the equation

$$a(n) = c_1 a(n-1) + c_2 a(n-2), \text{ for all } n \geq 2.$$

$\square$

Example.

$$a(n) = a(n-1) + 6a(n-2)$$

The characteristic polynomial is

$$x^2 - x - 6$$

With roots 3 and -2.

One root is $r = 3$; picking $\theta = 1$, we get a sequence

$$\begin{aligned} & \left\langle 3^0, 3^1, 3^2, 3^4, \ldots \right\rangle \\ = \ & \langle 1, 3, 9, 27, \ldots \rangle \end{aligned}$$

So if the base cases are $a(0) = 1$ and $a(1) = 3$, we should pick $r = 3$ and $\theta = 1$.

Trying the other root $r = -2$ and picking $\theta = 3$ we get

$$\left\langle 3 \cdot (-2)^0, 3 \cdot (-2)^1, 3 \cdot (-2)^2, 3 \cdot (-2)^3, ... \right\rangle$$
$$\langle 3, -6, 12, -24, ... \rangle$$

So if the base cases are $a(0) = 3$ and $a(1) = -6$, we should pick $r = -2$ and $\theta = 3$.

Unfortunately the base cases may not allow such a simple solution.

Consider

$$\begin{aligned} a(0) &= -2 \\ a(1) &= 3 \\ a(n) &= a(n-1) + 6a(n-2) \end{aligned}$$

The roots are $r_1 = 3$ and $r_2 = -2$. Trying r_1 we must find θ such that

$$\theta r_1^0 = -2 \text{ and } \theta r_1^1 = 3$$

Trying r_2 we must find a θ such that

$$\theta r_2^0 = -2 \text{ and } \theta r_2^1 = 3$$

Neither root gives a solution that agrees with both base cases.

Linear combinations of solutions

Suppose that we have a recurrence relation:

$$a(n) = c_1 a(n-1) + c_2 a(n-2), \text{ for all } n \geq 2$$

Now suppose we know a particular sequence w solves the recurrence. I.e.

$$w(n) = c_1 w(n-1) + c_2 w(n-2), \text{ for all } n \geq 2$$

Then, for any constant θ, we can define a new sequence y defined by

$$y(n) \triangleq \theta \times w(n), \text{ for all } n \in \mathbb{N}$$

This too will be a solution as

$$\begin{aligned} y(n) &= \theta \times w(n) \\ &= \theta(c_1 w(n-1) + c_2 w(n-2)) \\ &= c_1 \theta w(n-1) + c_2 \theta w(n-2) \\ &= c_1 y(n-1) + c_2 y(n-2), \text{ for all } n \geq 2 \end{aligned}$$

So multiplying a solution by any constant gives a solution.

Suppose that w and x are two solutions. I.e.

$$w(n) = c_1 w(n-1) + c_2 w(n-2), \text{ for all } n \geq 2$$

and

$$x(n) = c_1 x(n-1) + c_2 x(n-2), \text{ for all } n \geq 2$$

Then we can build a sequence z defined by

$$z(n) \triangleq w(n) + x(n), \text{ for all } n \in \mathbb{N}$$

Now z too will be a solution as

$$\begin{aligned} z(n) &= w(n) + x(n) \\ &= c_1 w(n-1) + c_2 w(n-2) + c_1 x(n-1) + c_2 x(n-2) \\ &= c_1(w(n-1) + x(n-1)) + c_2(w(n-2) + x(n-2)) \\ &= c_1 z(n-1) + c_2(z(n-2), \text{ for all } n \geq 2 \end{aligned}$$

So given any two solutions, x and w , any linear combination of them

$$z(n) \triangleq \theta_1 w(n) + \theta_2 x(n), \text{ for all } n \in \mathbb{N}$$

will also be a solution.

A set that is closed under linear combinations is called a *vector space*. If the degree is 2, all solutions can be formed as linear combination of just 2 (appropriately chosen) solutions. If there are two different roots, we have two solutions r_1^n and r_2^n, which will suffice.

Using both roots at once

Consider, again, the LHRRCC of degree 2.

$$a(n) = c_1 a(n-1) + c_2 a(n-2), \text{ for all } n \geq 2 \qquad (*)$$

with characteristic polynomial

$$x^2 - c_1 x - c_2 = 0$$

Let r_1 and r_2 be the roots of this polynomial.

Consider any two constants θ_1 and θ_2.

$$\theta_1 r_1^n + \theta_2 r_2^n \text{ is a solution to (*)}$$

since it is a linear combination of the solutions r_1^n and r_2^n.

Here is a direct proof:

$$\theta_1 r_1^n + \theta_2 r_2^n \text{ is a solution to (*)}$$

iff

$$\theta_1 r_1^n + \theta_2 r_2^n = c_1 \left(\theta_1 r_1^{n-1} + \theta_2 r_2^{n-1}\right) + c_1 \left(\theta_1 r_1^{n-2} + \theta_2 r_2^{n-2}\right)$$

iff

$$\theta_1 \left(r_1^n - c_1 r_1^{n-1} - c_2 r_1^{n-2}\right) + \theta_2 \left(r_2^n - c_1 r_2^{n-1} - c_2 r_2^{n-2}\right) = 0$$

iff

$$\theta_1 r_1^{n-1} \left(r_1^2 - c_1 r_1 - c_2\right) + \theta_2 r_2^{n-1} \left(r_2^2 - c_1 r_2 - c_2\right) = 0$$

iff

$$T$$

This proves the following theorem

Theorem: Given the LHRRCC of degree 2

$$a(n) = c_1 a(n-1) + c_2 a(n-2)., \text{ for all } n > 2 \qquad (*)$$

If r_0 and r_1 are the two roots of the characteristic polynomial

$$x^2 - c_1 x - c_2$$

then for any θ_1 and θ_2

$$\theta_1 r_1^n + \theta_2 r_2^n$$

is a solution to (*).

Furthermore, if we know $a(0)$ and $a(1)$.Then

$$\begin{aligned} \theta_1 + \theta_2 &= a(0) \\ \theta_1 r_1 + \theta_2 r_2 &= a(1) \end{aligned}$$

With 2 linear equations in 2 unknowns we can solve for θ_1 and θ_2 .

This will succeed provided $r_0 \neq r_1$.

So, when the roots are distinct, we can find the unique values for θ_1 and θ_2 that satisfy the two base cases.

Example:

$$\begin{aligned} a(0) &= -2 \\ a(1) &= 3 \\ a(n) &= a(n-1) + 6a(n-2) \end{aligned}$$

The characteristic polynomial

$$x^2 - x - 6 = 0$$

has roots $r_1 = 3$ and $r_2 = -2$. So we are looking for a solution of the form

$$\theta_1 3^n + \theta_2 (-2)^n$$

We have

$$\begin{aligned} \theta_1 + \theta_2 &= a(0) = -2 \\ 3\theta_1 - 2\theta_2 &= a(1) = 3 \end{aligned}$$

So

$$\begin{aligned} 3\theta_1 - 2(-2 - \theta_1) &= 3 \\ 5\theta_1 + 4 &= 3 \\ \theta_1 &= \frac{-1}{5} \end{aligned}$$

and

$$\begin{aligned} \theta_2 &= -2 - \theta_1 \\ \theta_2 &= \frac{-9}{5} \end{aligned}$$

Procedure

- From the RR derive the characteristic polynomial
- Find roots $r_1$and r_2 of the characteristic polynomial.
- If $r_1 \neq r_2$ then look for a solution of the solution of the form
$$\theta_1 r_1^n + \theta_2 r_2^n$$
- use the base cases to solve for θ_1 and θ_2

Example:

$$\begin{aligned} \mathit{fib}(0) &= 1 \\ \mathit{fib}(1) &= 1 \\ \mathit{fib}(n) &= \mathit{fib}(n-1) + \mathit{fib}(n-2) \end{aligned}$$

Form the characteristic polynomial

$$x^2 - x - 1 = 0$$

Find roots using the quadratic equation. The roots are

$$\frac{1+\sqrt{5}}{2} = 1.61803\cdots = \phi$$

and

$$\frac{1-\sqrt{5}}{2} = -0.618\,03\cdots = 1-\phi = \frac{-1}{\phi}$$

We are looking for a solution of the form

$$\theta_1\phi^n + \theta_2(1-\phi)^2$$

Using $\mathit{fib}(0) = \mathit{fib}(1) = 1$ we get

$$\begin{aligned} \theta_1 + \theta_2 &= 1 \\ \theta_1\phi + \theta_2(1-\phi) &= 1 \end{aligned}$$

Substitute $\theta_2 = 1 - \theta_1$ into $\theta_1\phi + \theta_2(1-\phi) = 1$ to get

$$\theta_1\phi + (1-\theta_1)(1-\phi) = 1$$

Now solve for θ_1.

$$\begin{aligned} \theta_1\phi + (1-\theta_1)(1-\phi) &= 1 \\ \theta_1\phi + (1-\phi) - \theta_1(1-\phi) &= 1 \\ \theta_1(\phi - (1-\phi)) &= \phi \\ \theta_1(2\phi - 1) &= \phi \\ \theta_1\sqrt{5} &= \phi \\ \theta_1 &= \frac{\phi}{\sqrt{5}} = \frac{1+\sqrt{5}}{2\sqrt{5}} = \frac{5+\sqrt{5}}{10} \end{aligned}$$

And then solve for θ_2

$$
\begin{aligned}
\theta_2 &= 1 - \theta_1 \\
&= \frac{\sqrt{5}}{\sqrt{5}} - \frac{\phi}{\sqrt{5}} \\
&= \frac{\sqrt{5} - \phi}{\sqrt{5}} \\
&= \frac{\frac{2\sqrt{5}}{2} - \frac{1+\sqrt{5}}{2}}{\sqrt{5}} \\
&= \frac{\frac{\sqrt{5}-1}{2}}{\sqrt{5}} \\
&= -\frac{1 - \phi}{\sqrt{5}}
\end{aligned}
$$

So the solution is

$$
\begin{aligned}
\textit{fib}(n) &= \frac{\phi}{\sqrt{5}}\phi^n - \frac{(1-\phi)}{\sqrt{5}}(1-\phi)^n \\
&= \frac{1}{\sqrt{5}}\phi^{n+1} - \frac{1}{\sqrt{5}}(1-\phi)^{n+1}
\end{aligned}
$$

$\square$

These ideas generalize to degrees larger than 2.

Repeated roots for LHRRCCs with degree 2

When $r_1 = r_2$ then $\theta_1 r_1^n + \theta_2 r_2^n$ can be written as θr^n where $\theta = \theta_1 + \theta_2$ and $r = r_1 = r_2$.

So the 2 base cases may form an overdetermined system: 2 equations and one unknown.

Example

$$\begin{aligned} a(0) &= 1 \\ a(1) &= 2 \\ a(n) &= 6a(n-1) - 9a(n-2), \text{ for } n \geq 2 \end{aligned}$$

The characteristic polynomial is

$$x^2 - 6x + 9$$

with root $r = 3$.

We look for a solution of the form

$$\theta r^n$$

But

$$\begin{aligned} \theta &= 1 \\ \theta r &= 2 \end{aligned}$$

is not solvable!□

Consider a quadratic with a repeated root

$$x^2 - 4x + 4$$

so $r = 2$. This is characteristic of the LHRRCC

$$a(n) = 4a(n-1) - 4a(n-2)$$

adding a couple of base cases $a(0) = 0$ and $a(1) = 2$ we get

n	$a(n)$
0	0
1	2
2	$4 \times 2 = 8$
3	$4 \times 8 - 4 \times 2 = 24$
4	$4 \times 24 - 4 \times 8 = 64$
5	$4 \times 64 - 4 \times 24 = 160$

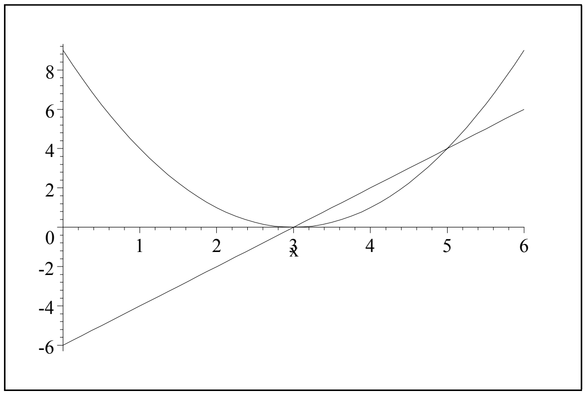
The solution is apparently $n2^n$.

This suggests nr^n is a potential solution in general.

Note that nr^n is $r \times nr^{n-1}$ and that nr^{n-1} is the derivative of our basic solution r^n.

So we might do well to look at derivatives.

When a polynomial has a repeated root, that root will also be a root of its derivative:



Consider $x^2 - 6x + 9 = (x-3)^2$ and its derivative $2(x-3)$, 3 is a root of both.

In general (for all $r \in \mathbb{R}$) if

$$p(x) = x^2 - c_1 x - c_2 = (x - r)^2$$

then r will also be a root of $p'(x) = 2x - c_1 = 2(x - r)$.

Define $p_0(x) \triangleq x^{n-2} \cdot p(x)$

r will also be a root of p_0 since

$$p_0(r) = r^{n-2} \cdot p(r) = 0.$$

r will be a root of p_0' since

$$p_0'(r) = r^{n-2} \cdot p'(r) + (n-2)r^{n-3} \cdot p(r) = 0.$$

Theorem: If a LHRRCC of the form

$$a(n) = c_1 a(n-1) + c_2 a(n-1), \text{ for all } n \geq 2 \qquad (*)$$

has a characteristic polynomial with one root r then,

$$nr^n$$

is a solution:

Proof: Let p and p_0 be polynomials defined by

$$\begin{aligned} p(x) &\triangleq x^2 - c_1 x - c_2 \\ p_0(x) &\triangleq x^{n-2} \cdot p(x) \end{aligned}$$

As we just saw, since r is the sole root of p, it is also a root of p_0'.

What is p_0'?

$$\begin{aligned} p_0(x) &= x^{n-2} \cdot p(x) = x^n - c_1 x^{n-1} - c_2 x^{n-2} \\ p_0'(x) &= n x^{n-1} - c_1(n-1)x^{n-2} - c_2(n-2)x^{n-3} \end{aligned}$$

Now

$$nr^n \text{ is a solution of the RR (*)}$$

if (for all $n \geq 2$)

$$nr^n = c_1(n-1)r^{n-1} + c_2(n-2)r^{n-2}$$

if (for all $n \geq 2$)

$$nr^n - c_1(n-1)r^{n-1} - c_2(n-2)r^{n-2} = 0$$

if (for all $n \geq 2$)

$$r \cdot (nr^{n-1} - c_1(n-1)r^{n-2} - c_2(n-2)r^{n-3}) = 0$$

if (for all $n \geq 2$)

$$r \cdot p_0'(r) = 0,$$

which is true as r is a root of p_0'.□

Theorem: If a LHRRCC of the form

$$a(n) = c_1 a(n-1) + c_2 a(n-1), \text{ for all } n \geq 2 \qquad (*)$$

has a characteristic polynomial with one root r then, for any $\alpha_0, \alpha_1 \in \mathbb{R}$, $(\alpha_0 + \alpha_1 n)r^n$ is a solution.

Proof: This is just a linear combination of the solutions r^n and nr^n.□

We can use the base cases to compute the α_0 and α_1.

Back to the earlier example

$$\begin{aligned} a(0) &= 1 \\ a(1) &= 2 \\ a(n) &= 6a(n-1) - 9a(n-2), \text{ for } n \geq 2 \end{aligned}$$

The root of the characteristic polynomial $x^2 - 6x + 9$ is 3.

From the theorem, the solution is of the form $(\alpha_0 + \alpha_1 n) \cdot 3^n$

Now solve

$$\begin{aligned} \alpha_0 &= 1 \\ (\alpha_0 + \alpha_1) \cdot 3 &= 2 \end{aligned}$$

so

$$\alpha_1 = \frac{2}{3} - 1 = -\frac{1}{3}$$

Check:

n	$a(n)$	$(1 - \frac{1}{3}n) \cdot 3^n$
0	1	$(1 - \frac{1}{3} \cdot 0) \cdot 3^0 = 1.0$
1	2	$(1 - \frac{1}{3} \cdot 1) \cdot 3^1 = 2.0$
2	$6 \times 2 - 9 \times 1 = 3$	$(1 - \frac{1}{3} \cdot 2) \cdot 3^2 = 3.0$
3	$6 \times 3 - 9 \times 2 = 0$	$(1 - \frac{1}{3} \cdot 3) \cdot 3^3 = 0$
4	$6 \times 0 - 9 \times 3 = -27$	$(1 - \frac{1}{3} \cdot 4) \cdot 3^4 = -27.0$
5	$6 \times -27 - 9 \times 0 = -162$	$(1 - \frac{1}{3} \cdot 5) \cdot 3^5 = -162.0$

Procedure for LHRRCC of degree 2

- From the RR derive the characteristic polynomial
- Find roots $r_1$and r_2 of the characteristic polynomial.
- If $r_1 \neq r_2$ then look for a solution of the solution of the form

$$\theta_1 r_1^n + \theta_2 r_2^n$$

  * Use the base cases to solve for θ_1 and θ_2.
- If the sole root is r then look for a solution of the form

$$(\alpha_0 + \alpha_1 n) r^n$$

  * Use the base cases to solve for α_0 and α_1.

One can generalize these theorems and the resulting procedure to LHRRCCs with any degree and any number of repeated roots. See Gossett's book.