Quiz 2 – Solution

Engineering 3422, 2004

Wednesday Nov 10, 2004

Name Solution
Q0 [12]
Consider a one-way infinite sequence defined as follows

$$\begin{aligned} &a(0) = 0 \\ &a(n) = 4 \times a(n/2), \text{ if } n \geq 2 \text{ and } n \text{ is even} \\ &a(n) = a(n-1) + 2 \times n - 1, \text{ if } n \geq 1 \text{ and } n \text{ is odd} \end{aligned}$$

Use *the principle of complete induction* to prove that $\forall n \in \mathbb{N},\ a(n) = n^2$.

(a)[2] What must be proved in the base step (only one base step is needed)

- *That* $a(0) = 0^2$
- *Comment: A number of people forgot that* 0 *is a natural number, either in part (a), part (c), or both.*

(b)[2] Prove the base step.

- $$\begin{aligned} & a(0) \\ = \ & 0 \text{ By defn of } a \\ = \ & 0^2 \end{aligned}$$

(c)[4] What must be proved in the inductive step? Be clear about exactly how variables are quantified.

- *That, for all* k *in* $\{1, 2, ...\}$*, if, for all* $j \in \{0, 1, ..., k-1\}$*,* $a(j) = j^2$ *then* $a(k) = k^2$

- *Comment: A lot of people wanted to use $a(k+1) = (k+1)^2$ as the consequent. This is okay, provided you make suitable changes to the rest of the statement. The alternative answer is: That for all k in $\mathbb{N}$, if for all $j \in \{0, 1, ..., k\}$, $a(j) = j^2$ then $a(k+1) = (k+1)^2$*

- *Comment: Some people used $P(k)$ instead of $a(k) = k^2$. This is fine* provided *you had defined P.*

- *Comment: A few people answered in the form of a recipe for proving the what must be proved. E.g.:*

  - *Assume that $k \in \{1, 2, ...\}$*
  - *Assume that for all $j \in \{0, 1, ..., k-1\}$, $a(j) = j^2$*
  - *Show that $a(k) = k^2$*

  This answers a slightly different question.

(d)[4] Prove the inductive step, being careful to clearly state the induction hypothesis and to clearly indicate where the induction hypothesis is being used.

- *Let k be any natural number ≥ 1*

- *Assume, as the induction hypothesis, that $\forall j \in \{0, 1, ..., k-1\}, a(j) = j^2$*

- *It remains to show $a(k) = k^2$.*

- *Case: k is even*

  - *As k is even and greater ≥ 1, $k/2$ is in $\{0, 1, ..., k-1\}$, so by the induction hypothesis, $a(k/2) = (k/2)^2$*
  - $$\begin{aligned} a(k) &= 4 \times \underline{a(k/2)} \text{ By defn of } a \\ &= 4 \times \overline{(k/2)^2} \text{ By ind. hyp.} \\ &= k^2 \text{ Algebra} \end{aligned}$$

- *Case: k is odd*

- *As* $k \geq 1$, $k-1$ *is in* $\{0, 1, ..., k-1\}$, *so by the induction hypothesis,* $a(k-1) = (k-1)^2$
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$$\begin{aligned} a(k) &= \underline{a(k-1)} + 2k - 1 \text{ By defn of } a \\ &= (k-1)^2 + 2k - 1 \text{ By ind. hyp.} \\ &= k^2 - 2k + 1 + 2k - 1 \text{ Algebra} \\ &= k^2 \text{ Algebra} \end{aligned}$$

Q1 [9]

Suppose you want to prove that any finite set S, such that $|S| \geq 3$, has $|S|\,(|S|-1)(|S|-2)/6$ subsets of size 3, using simple induction.

(a)[3] State the property that must be proved for all $n \in \{3, 4, ...\}$. Be careful to properly quantify all variables.

$P(n)$ is:

- *For all sets S of size n, S has $n(n-1)(n-2)/6$ subsets of size 3.*
- Comment: It is very important that P be defined as a function from the naturals to the booleans. A number of people defined $P(n)$ to be a number, which does not make sense. It is important to pay attention to the types of things. Sets, booleans, and numbers are different types; so you don't want to use a number where a set is needed, or a set where a boolean is needed.

(b)[3] Without reference to P, state what must be proved in the base step. Be clear about exactly how variables are quantified.

- *For all sets S of size 3, S has 1 subset of size 3.*

(c)[3] Without reference to P, state what must be proved in the inductive step. Be clear about exactly how variables are quantified.

- *For all $k \in \{3, 4, ...\}$,*
  - *if all sets of size k have $k(k-1)(k-2)/6$ subsets of size 3,*
  - *then all sets of size $k+1$ have $(k+1)\,k(k-1)/6$ subsets of size 3.*
