Engi 3422 Final Exam

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Instructions: Answer all questions. Write your answers in the space provided. Request a yellow booklet if more space is required. This is an **closed** book test, no textbooks or notes permitted. Calculators are permitted if they are nonprogrammable and can not store text or formulas. Dictionaries are permitted.

Total points: 100

Name:

Student #:

Q0 [12]

(a) Label each of the following predicate expressions, describing a relation R, with the adjective it defines. Select your answers from the set

{onto, one-one, partial function, function, none of the others}

1. $\forall x_0 \in \mathrm{dom}(R), \forall x_1 \in \mathrm{dom}(R), \forall y \in \mathrm{rng}(R), x_0 R y \wedge x_1 R y \rightarrow x_0 = x_1$

2. $\forall y_0 \in \mathrm{rng}(R), \forall y_1 \in \mathrm{rng}(R), \forall x \in \mathrm{dom}(R), x R y_0 \wedge x R y_1 \rightarrow y_0 = y_1$

3. $\forall x \in \mathrm{dom}(R), \exists y \in \mathrm{rng}(R), x R y$

4. $\forall y \in \mathrm{rng}(R), \exists x \in \mathrm{dom}(R), x R y$

5. $\forall x \in \mathrm{dom}(R), \forall y \in \mathrm{rng}(R) \cap \mathrm{dom}(R), \forall z \in \mathrm{rng}(R), x R y \wedge y R z \rightarrow x R z$

(b) What subset of the above properties {1,2,...,5} must be conjoined (joined by an "and") to give the definition of a function.

Q1 [8]

Classify each of the following propositional statements as one of: tautology, contradiction, conditional statement.

- $(P \rightarrow Q) \leftrightarrow (\neg P \vee Q)$

- $(P \rightarrow Q) \leftrightarrow (\neg P \wedge Q)$

- $(P \rightarrow Q) \leftrightarrow (P \wedge \neg Q)$

- $(P \rightarrow Q) \wedge (\neg P \vee Q)$

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Q2 [12]
Consider a one-way infinite sequence defined as follows

$$
\begin{aligned}
a(0) &= 0 \\
a(n) &= 4 \times a(n/2), \text{ if } n \geq 1 \text{ and } n \text{ is even} \\
a(n) &= a(n-1) + 2 \times n - 1, \text{ if } n \geq 1 \text{ and } n \text{ is odd}
\end{aligned}
$$

Use *the principle of complete induction* to prove that $\forall n \in \mathbb{N}$, $a(n) = n^2$.

(a) What must be proved in the base step (only one base step is needed)

(b) Prove the base step.

(c) What must be proved in the inductive step? Be clear about exactly how variables are quantified.

(d) Prove the inductive step, being careful to clearly state the induction hypothesis and to clearly indicate where the induction hypothesis is being used.

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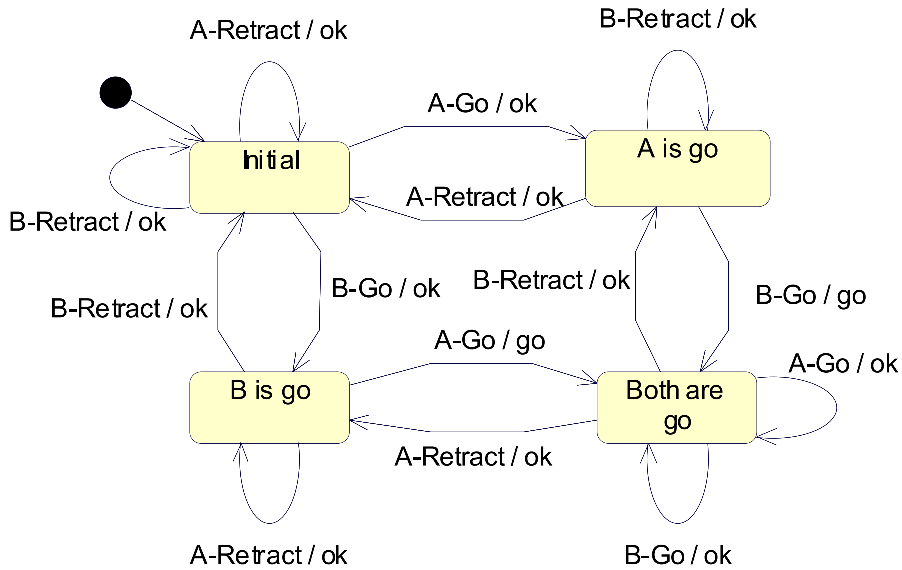
Q3 [4]
Consider the finite state automaton described below.



Consider a directed walk with vertex sequence

(Initial, A is go, Both are go, A is go, Both are go)

What behaviour is associated with this walk?

Q4 [8]
Bob wrote the following line of C code

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if( (a >= b && c >= d) || ( a < b && c < d ) )
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Alice suggested changing it to

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if( (a < b) ==(c < d) )
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but Bob is not convinced. Give an algebraic proof of the following equivalence. Give hints for each step.

$$P \leftrightarrow Q \Leftrightarrow (\neg P \wedge \neg Q) \vee (P \wedge Q)$$

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Q5 [10]

The Cheap Date Theatre wishes to put on a play with 5 scenes and 6 characters. Each character must be played by exactly one actor, and no actor may play two characters who appear in the same scene; otherwise any actor may play any character. The producer would like to know the minimum number of actors required. The scenes are as follows

Scene 1: Abigail, Bertram, Christophe
Scene 2: Bertram, Christophe, Dehlila
Scene 3: Abigail, Bertram, Edgar
Scene 4: Abigail, Christophe, Edgar
Scene 5: Bertram, Edgar, Ferdinand

(a) Use your knowledge of graph theory to find the minimal number of actors required. Demonstrate that this number is sufficient using a graph.

(b) Explain why fewer actors can not be used. Use a graph to illustrate.

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Q6 [8]

For sets A and B define $A\Delta B = \left(A \cap \overline{B}\right) \cup \left(B \cap \overline{A}\right)$. Prove that this is an associative operator. That is

$$A\Delta(B\Delta C) = (A\Delta B)\Delta C$$

Give hints for each step.

Q7 [6]

Suppose that $G = (V, E, \phi)$ is a *simple* graph. Let $n = |V|$. Let the degree of each vertex, v, be defined as

$$\deg(v) = \sum_{e \in E} \begin{cases} 0 & \text{if } v \notin \phi(e) \\ 1 & \text{if } v \in \phi(e) \text{ and } e \text{ is not a loop} \\ 2 & \text{if } v \in \phi(e) \text{ and } e \text{ is a loop} \end{cases}$$

Let $D = \{d \mid d = \deg(v), \text{ for some } v \in V\}$ be the set of all degrees of vertices in G.

Since G is simple, $D \subseteq \{0, 1, ..., n-1\}$

Prove that if $n > 1$ then $|D| < n$.

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Q8 [12]

Let G be a set of generating stations, S be a set of substations, and D be a set of distribution stations.

Let R be a relation with domain and range $G \cup S \cup D$, and xRy meaning that power can be transmitted directly from x to y.

Express the following sets or predicates. Make sure that the free variables are correct.

- The set of all substations in S that can receive power directly from generating station x.
- No distribution station in D can receive power directly from any generating station in G
- All generating stations in G can send power directly to at least 2 substations in S.
- Let xR^*y mean $\exists i \in \mathbb{N}, xR^iy$. Express the set of all distribution stations that can not receive power directly or indirectly from any generating station.

Q9 [8]

(a) Derive a general solution to the following recurrence relation

$$a(n) = 6a(n-1) - 9a(n-2), \text{ for all } n \geq 2$$

(b) Give a solution to the system formed by the above recurrence relation and the following base cases

$$a(0) = 2, \qquad a(1) = 18$$

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Q10 [12]
Classify the following claims as true or false

- $a \mid b$ implies $b \mid a$, for all $a, b \in \{1, 2, ...\}$
- If $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$ then $a \equiv c \pmod{m}$, for all $a, b, c \in \mathbb{Z}$, and $m \in \{1, 2, ...\}$
- $\gcd(2^{100} \cdot 3^{200} \cdot 5^{300}, 2^{200} \cdot 3^{100}) = 2^{100} \cdot 3^{200}$
- For any numbers $a \in \mathbb{N}$ and $b \in \{1, 2, 3, ...\}$ there exists a unique pair $(q, r) \in \mathbb{Z} \times \mathbb{Z}$ such that

$$a = bq + r$$

- Every positive natural number has a unique prime decomposition.
- For every set S and predicate P

$$(\forall x \in S, P) \rightarrow (\exists x \in S, P)$$

Bonus [5]
How many zeros appear at the end of 100!. [Hint. Think about the prime decompositions.]

Best of luck on your remaining exams. Happy holidays