Unit 1. Propositional Logic

Reading — do all quick-checks

Propositional Logic: Ch. 2.intro, 2.2, 2.3, 2.4.

Review 2.9

Statements or propositions

Defn: A **statement** is an assertion that may be labelled true or false.

Defn: **Proposition** is another word for statement.

Examples:

The following are propositions

1. $\sqrt{2} > 1$ — true
2. all planar graphs are 4-colourable — true
3. $6 \times 9 = 42$ — false
4. the square root of 2 is rational — false
5. every even integer greater than 2 is the sum of two primes. — unknown
6. the equation $x^2 + 1 = 0$ has no real root — true

In the following propositions, the truth or falsity of the statement depends on something unknown. Nevertheless, we will accept them as propositions

1. i is the sum of two primes — the truth or falsity of this statement explicitly depends on the value of i
2. $x^2 = x$ — the truth or falsity of this statement explicitly depends on the value of x.

3. if x is 0 or 1 then $x^2 = x$ — “formally” this statement depends on the value of x, even though it is, in a sense, necessarily true.
4. The tide is high — the truth or falsity of this statement implicitly depends on the time of day and the location.
5. Wire “a” has a high voltage — the voltage on the wire may vary with time, so the truth or falsity of this statement may depend implicitly on the time.

Counterexamples:

1. $\sqrt{2}$ — this is a number, not a statement
2. the prime numbers — this is a set, not a statement
3. is the sum of two primes — this is a predicate, not a statement

Truth values

All true propositions are logically equivalent, as are all false propositions.

- We use the symbol F to represent any false proposition
- We use the symbol T to represent any true proposition

Alternative notations

This course	**Digital Logic**	**C++/Java**
F	0	false
T	1	true

Compound Propositions

AND, OR, and NOT

Aside: An **algebra** consists of a set of values and a set of operations than operate on that set.

F and T are the values of a simple algebra called **propositional algebra** or **propositional calculus**.

We will use P, Q, and R as variables that range over the values F and T.

Just as $+$, $-$, $\times$ and $\div$ combine numerical expressions, we have algebraic operations that combine propositional expressions.

Propositional operator **AND (conjunction)**: $P \wedge Q$ is T if and only if both P and Q are T.

The operands are called **conjuncts**.

P	Q	$P \wedge Q$
F	F	☐
F	T	☐
T	F	☐
T	T	☐

Example compound proposition:

all planar graphs are 4-colourable $\land 6 \times 9 = 42$

This evaluates to F

. . . .

Defn: We write $A \Leftrightarrow B$ to mean two propositional expressions A and B are equal regardless of the truth values assigned to their propositional variables. We say that expressions A and B are **logically equivalent.**

(N.B. $\Leftrightarrow$ is a relation between propositional expressions.)

Let's check $P \wedge Q \Leftrightarrow Q \wedge P$:

- Assigning F to P and F to Q we get $F \wedge F$ on the left and $F \wedge F$ on the right
- Assigning F to P and T to Q we get $F \wedge T$ on the left and $T \wedge F$ on the right
- Assigning T to P and F to Q we get $T \wedge F$ on the left and $F \wedge T$ on the right
- Assigning T to P and T to Q we get $T \wedge T$ on the left and $T \wedge T$ on the right

In each case, the left and the right values are the same.

We conclude that $P \wedge Q \Leftrightarrow Q \wedge P$ is an **algebraic law**.

Defn: For propositional expressions A and B, we write $A \Rightarrow B$ to mean that for each assignment to the propositional variables such that A evaluates to true, B also evaluates to true. We say that B **can be inferred from** A.

Let's check $P \wedge Q \Rightarrow P$:

- Assigning F to P and F to Q we get $F \wedge F$ on the left and F on the right
- Assigning F to P and T to Q we get $F \wedge T$ on the left and F on the right
- Assigning T to P and F to Q we get $T \wedge F$ on the left and T on the right
- Assigning T to P and T to Q we get $T \wedge T$ on the left and T on the right

In each case, when the expression on the left evaluates to true, the expression on the right is also true.

We conclude that $P \wedge Q \Rightarrow P$ is an **algebraic law**.

Note: We will define **propositional expression**, **equivalence**, and **inference** more formally later.

……

Some algebraic laws about AND

Identity : $P \wedge T \Leftrightarrow P$

Domination : $P \wedge F \Leftrightarrow F$

Idempotence : $P \wedge P \Leftrightarrow P$

Commutativity : $P \wedge Q \Leftrightarrow Q \wedge P$

Associativity : $P \wedge (Q \wedge R) \Leftrightarrow (P \wedge Q) \wedge R$

Because of the associativity law, we will write $P \wedge Q \wedge R$ without parentheses.

Propositional operator **OR (disjunction):** $P \vee Q$ is T if and only if P is T or Q is T, or both are T.

The operands are called **disjuncts**

P	Q	$P \vee Q$
F	F	☐
F	T	☐
T	F	☐
T	T	☐

Example: today is sunny $\vee$ today I have an Umbrella

Some algebraic laws about OR

Identity: $P \vee F \Leftrightarrow P$

Domination: $P \vee T \Leftrightarrow T$

Idempotence: $P \vee P \Leftrightarrow P$

Commutativity: $P \vee Q \Leftrightarrow Q \vee P$

Associativity: $P \vee (Q \vee R) \Leftrightarrow (P \vee Q) \vee R$

Some laws about AND and OR

Distributivity of AND over OR:

$$P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R)$$

Distributivity of OR over AND:

$$P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee R)$$

Precedence:

- Note that $P \wedge Q \vee R$ might be interpreted as $P \wedge (Q \vee R)$ or $(P \wedge Q) \vee R$. These expressions are not logically equivalent.
- Consider this English ‘sentence’: The court finds that you must serve 90 days and pay a $1000 fine or say you are really very very sorry.
- Usually (i.e. in digital logic, in most programming languages, and in most mathematical papers and

books) AND has higher precedence than OR. Thus $P \wedge Q \vee R$ is usually interpreted as $(P \wedge Q) \vee R$.

- However, in this course, we follow the text and always use parentheses when mixing the $\wedge$ operator with the $\vee$ operator.

Propositional operator **NOT (negation):** $\neg P$ is T if and only if P is F

P	$\neg P$
F	☐
T	☐

Precedence: NOT has higher precedence than AND and OR. E.g. we interpret $\neg P \vee Q$ as meaning $(\neg P) \vee Q$.

A law about NOT

Involution: $\neg\neg P \Leftrightarrow P$

Some laws about NOT, AND, and OR:

De Morgan's law: $\neg(P \wedge Q) \Leftrightarrow \neg P \vee \neg Q$

De Morgan's law: $\neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q$

Contradiction: $\neg P \wedge P \Leftrightarrow F$

Excluded Middle: $\neg P \vee P \Leftrightarrow T$

Alternative notations

Math	**Digital Logic**	**C++/Java**	**C++/Java (bitwise)**
F	0	false	0
T	1	true	-1
$\wedge$	$\cdot$	&&	&
$\vee$	$+$	||	|
$\neg$		!	~

Showing two sentences equivalent

Truth tables method

We can verify the laws using the "method of truth tables". Example

$$\neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q$$

There are 4 possible values for P, and Q.

We make a table and work out the value of each compound sentence

P	Q	$(P \vee Q)$	$\neg(P \vee Q)$	$\neg P$	$\neg Q$	$\neg P \wedge \neg Q$
F	F	F	T	T	T	T
F	T	T	F	T	F	F
T	F	T	F	F	T	F
T	T	T	F	F	F	F

Note that the columns for $\neg(P \vee Q)$ and $\neg P \wedge \neg Q$ are the same.

So $\neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q$ is a law.

Note that the number of rows is 2^n where n is the number of variables.

E.g. with 7 variables, 128 rows. With 10 variables, 1024 rows.

Algebraic method

We can apply the laws to create new laws.

$$
\begin{array}{lll}
 & (P \vee Q) \wedge R & \\
\Leftrightarrow & R \wedge (P \vee Q) & \text{Commutativity} \\
\Leftrightarrow & (R \wedge P) \vee (R \vee Q) & \text{Distributivity of AND over OR} \\
\Leftrightarrow & (P \wedge R) \vee (Q \wedge R) & \text{Commutativity (twice)}
\end{array}
$$

This shows

Distributivity of AND over OR: $(P \vee Q) \wedge R \Leftrightarrow (P \wedge R) \vee (Q \wedge R)$

We also have:

Distributivity of OR over AND: $(P \wedge Q) \vee R \Leftrightarrow (P \vee R) \wedge (Q \vee R)$

More Operators

Biconditional and Implication

Propositional operator **BICONDITIONAL**: $P \leftrightarrow Q$ is T if and only if P and Q are both T or both F.

We can define $P \leftrightarrow Q$ by a law

Defn of biconditional: $P \leftrightarrow Q \Leftrightarrow (P \wedge Q) \vee (\neg P \wedge \neg Q)$

or by a table

P	Q	$P \leftrightarrow Q$
F	F	☐
F	T	☐
T	F	☐
T	T	☐

In English we say “if and only if” (abbreviated iff). Example:

- p is prime iff p has exactly two factors.
- *I wear my hat iff it snows.* This will be false only when
  - ∗ I wear my hat, but it is not snowing
  - ∗ I don’t wear my hat, but it is snowing

Transitivity : $(P \leftrightarrow Q) \wedge (Q \leftrightarrow R) \Rightarrow (P \leftrightarrow R)$
Reflexivity : $P \leftrightarrow P \Leftrightarrow T$
Commutativity : $P \leftrightarrow Q \Leftrightarrow Q \leftrightarrow P$

Question: How is ↔ different from ⇔ ?

- ↔ is a propositional operator. We use it to *combine* two propositional expressions to make a new propositional expression. So $P \leftrightarrow P \wedge Q$ is neither true nor false, it is a propositional expression.
- ⇔ is a relation on propositional expressions. We use it to *compare* two propositional expressions. For example $P \Leftrightarrow P \wedge Q$ is false, since the two expressions are not equivalent.

Propositional operator **IMPLICATION**: Suppose I say

- “If it is snowing, I wear my hat”

This will be false if and only if it snows and I don’t wear my hat.

We use the notation $P \rightarrow Q$ for an expression that is F only when P is T but Q is F.

It would be the same to say

- Either it isn't snowing or I wear my hat.

Another example:

- For all integers n,greater than 2, if n is prime, then n is odd.
- This means the same as: For all integers n,greater than 2, n is not prime or n is odd.

We can define $\rightarrow$ by the law

Defn of implication: $P \rightarrow Q \Leftrightarrow \neg P \vee Q$

or the table

P	Q	$P \rightarrow Q$
F	F	☐
F	T	☐
T	F	☐
T	T	☐

Example: x is prime $\rightarrow$ x is odd. (Note we take the primes to be 2, 3, 5, 7, ...)

- For $x = 0$ we have $F \rightarrow F$ so the statement is T
- For $x = 1$ we have $F \rightarrow T$ so the statement is T
- For $x = 2$ we have $T \rightarrow F$ so the statement is F

- For $x = 3$ we have $T \rightarrow T$ so the statement is T
- For $x = 4$ we have $F \rightarrow F$ so the statement is T

We can conclude that the statement may be T or F depending on the value of x.

There are many useful laws about implication

$$\begin{array}{rcl}
\text{Shunt} & : & P \rightarrow (Q \rightarrow R) \Leftrightarrow (P \wedge Q) \rightarrow R \\
\text{Contrapositive} & : & P \rightarrow Q \Leftrightarrow \neg Q \rightarrow \neg P \\
\text{Domination} & : & F \rightarrow P \Leftrightarrow T \\
\text{Domination} & : & P \rightarrow T \Leftrightarrow T \\
\text{Identity} & : & T \rightarrow P \Leftrightarrow P \\
\text{Anti-identity} & : & P \rightarrow F \Leftrightarrow \neg P \\
\text{Modus Ponens} & : & P \wedge (P \rightarrow Q) \Rightarrow Q \\
\text{Transitivity} & : & (P \rightarrow Q) \wedge (Q \rightarrow R) \Rightarrow (P \rightarrow R) \\
\text{Reflexivity} & : & P \rightarrow P \Leftrightarrow T \\
\text{Anti-symmetry} & : & (P \rightarrow Q) \wedge (Q \rightarrow P) \Leftrightarrow P \leftrightarrow Q
\end{array}$$

Precedence:

- The $\wedge$, $\vee$ and $\neg$ operators have higher precedence than $\rightarrow$ and $\leftrightarrow$.
- The $\rightarrow$ and $\leftrightarrow$ operators have the same precedence.

- I strongly suggest using parentheses when a $\rightarrow$ operator is mixed with another operator of the same precedence. E.g. $A \rightarrow B \leftrightarrow C$.

Aside: The English word “if” usually indicates some form of *causality*. John of Navarre once said (only in French)

“if my mother had been a man, then I would be king”

If we naively consider this “if” to be implication, then we can see it is true: His mother was not a man, he was not king, so we have $F \rightarrow F$ which is T. However the same analysis applies to the statement

“if my mother had been a peasant, then I would be king”

which John probably would have considered a false claim. In English, “if A then B” often means: ‘in any possible world where A is true, B is also true’. What makes John’s statement humorous is that we must consider all possible worlds in which his mother was a man. In any case, the English use of the word “if” is clearly more complex than implication. Implication is much simpler, meaning simply “not A, or B”.

XOR, NAND and NOR

These three operators are the negations of the BICONDITIONAL, AND, and OR

$$\begin{aligned}\text{Defn of XOR} &: P \oplus Q \Leftrightarrow \neg(P \leftrightarrow Q)\\ \text{Defn of NAND} &: P \overline{\wedge} Q \Leftrightarrow \neg(P \wedge Q)\\ \text{Defn or NOR} &: P \veebar Q \Leftrightarrow \neg(P \vee Q)\end{aligned}$$

P	Q	$P \oplus Q$	$P \overline{\wedge} Q$	$P \veebar Q$
F	F	F	T	T
F	T	T	T	F
T	F	T	T	F
T	T	F	F	F

Note that the English word OR has many different meanings:

- ‘Comes with fries or salad’: Comes with fries $\overline{\wedge}$ comes with salad.
- ‘The exam is tomorrow or the next day’: The exam is tomorrow $\oplus$ the exam is the next day.
- ‘Either the station is off air or my radio is broken’: The station is off air $\vee$ my radio is broken.

Tautology and equivalence

In this section and the next, we formalize the ideas of equivalence and proof.

Definition a *propositional expression* is an expression made up of

- the constants T and F
- any number of propositional variables P, Q, R, ...
- the operators $\wedge$, $\vee$, $\neg$, ...
- parentheses

Definitions

- A propositional expression is a **tautology** iff it evaluates to T regardless of the truth values assigned to its propositional variables.
- A propositional expression is a **contradiction** iff it evaluates to F regardless of the truth values assigned to its propositional variables.
- A propositional expression is a **conditional statement** otherwise.

Using these definitions we can give a new definition to the relation of **equivalence** ($\Leftrightarrow$)

Two propositional expression A and B are **equivalent** iff $A \leftrightarrow B$ is a tautology.

Examples:

- $P \vee Q \vee (\neg P \wedge \neg Q)$ is a $\boxed{\qquad\qquad\qquad}$
- $(P \vee Q) \wedge (\neg R \vee \neg Q) \wedge (\neg R \vee \neg P) \wedge R$ is a $\boxed{\qquad\qquad\qquad\qquad}$
- $P \wedge (\neg P \vee \neg Q)$ is a $\boxed{\qquad\qquad\qquad\qquad\qquad\qquad}$
- $P \wedge Q \Leftrightarrow \neg(\neg P \wedge \neg Q)$ does not hold because $P \wedge Q \leftrightarrow \neg(\neg P \wedge \neg Q)$ is not a tautology
- $P \wedge Q \Leftrightarrow \neg(\neg P \vee \neg Q)$ holds because $P \wedge Q \leftrightarrow \neg(\neg P \vee \neg Q)$ is a tautology
- $P \Rightarrow P \vee Q$ holds because $P \rightarrow (P \vee Q)$ is a tautology

Note:

- A propositional expression A is a tautology iff $A \Leftrightarrow T$.
- A propositional expression A is a tautology iff $T \Rightarrow A$
- If $A \Leftrightarrow B$ then B is a tautology iff A is a tautology.

Substitution Principles and Proof

Substitution principles

Principle: **Substituting an equivalent statement.** If $A \Leftrightarrow B$ and (A) is a component of an expression C then $C \Leftrightarrow D$ where D is obtained by replacing the (A) component of C by (B).

Note: In applying this principle, you can add and remove *redundant* parentheses at will.

Example: We know $P \vee T \Leftrightarrow T$. [This is the $A \Leftrightarrow B$] So in the statement $Q \wedge (P \vee T)$ [this is the C] we can replace $(P \vee T)$ by T to get $Q \wedge T$ [this is the D] . We conclude $Q \wedge (P \vee T) \Leftrightarrow Q \wedge T$.

Example: We know $\neg\neg P \Leftrightarrow P$ [This is the $A \Leftrightarrow B$] So in the conditional statement $\neg\neg P \vee \neg Q$ [this is the C] we substitute to get $P \vee \neg Q$ [this is the D]. We conclude

$$\neg\neg P \vee \neg Q \Leftrightarrow P \vee \neg Q$$

Notn: **Substitution notation.**

- Let A and B be propositional expressions, and V be a propositional variable.
- We will write $B[V := A]$ to mean the expression B with every occurrence of the variable V replaced by (A).

Example:

- $(P \wedge Q \wedge R)[Q := S \vee T]$ is the expression
- $(P \vee \neg P)[P := P \vee Q]$ is the expression

Note: Sometimes we want to simultaneously replace multiple variables. I'll use the notation $C[V, W := A, B]$.

Example: $(\neg P \vee Q)[P, Q := \neg P, \neg Q]$ is

……

Aside: Using the substitution notation, we can restate the 'principle of substituting an equivalent statement':

Principle: **Substituting an equivalent statement (restated).** If A, B, and C are propositional expressions, V is a propositional variable and $A \Leftrightarrow B$, then

$$C[V := A] \Leftrightarrow C[V := B]$$

……

Principle: **Replacing a logic variable in a tautology.** For any propositional expressions A, B, C and any propositional variable V:

- if B is a tautology then $B[V := A]$ is also a tautology; and
- if $B \Leftrightarrow C$ then $B[V := A] \Leftrightarrow C[V := A]$.

Notes:

- The second bullet follows from the first. Why?
- Again removing redundant parentheses is ok.
- This principle can be extended to simultaneous replacement of multiple variables.

Examples:

- Replacing P by $(P \vee Q)$ in the tautology $P \vee \neg P$ gives , so this too must be a tautology.
- We know that $(P \rightarrow Q) \Leftrightarrow (\neg P \vee Q)$ is an equivalence; simultaneously replacing P with $\neg P$ and Q with $\neg Q$ we get

Algebraic proof

We can use these principles to formalize the notion of a proof of equivalence.

Defn: An **algebraic proof** of an equivalence $A_0 \Leftrightarrow A_n$ is a sequence of statements written

$$\begin{array}{ll} & A_0 \\ \Leftrightarrow & A_1 \text{ hint}_0 \\ \Leftrightarrow & \ldots \\ \Leftrightarrow & A_n \text{ hint}_{n-1} \end{array}$$

where, for each i, $A_i \Leftrightarrow A_{i+1}$ can be seen to be an equivalence using the substitution and variable replacement principles and previously proved laws (and tautologies). The hint is used to indicate to the law used.

Convention: Whenever substitution is involved, I like to underline the part of the expression that is about to be substituted for. This makes the proof much easier to follow.

Here are some substitutions into laws

Def^n impl. $P \to Q \Leftrightarrow \neg P \vee Q$ $[P, Q := \neg P, \neg Q]$

after replacement:

Commutativity $P \vee Q \Leftrightarrow Q \vee P$ $[Q := \neg Q]$

after replacement:

Def^n impl. $P \to Q \Leftrightarrow \neg P \vee Q$ $[P, Q := Q, P]$

after replacement:

Example: Here is an algebraic proof of the contrapositive law using only laws presented earlier.

Proof. RTP $\neg P \to \neg Q \Leftrightarrow Q \to P$

$\neg P \to \neg Q$

$\Leftrightarrow$ $\underline{\neg\neg P} \vee \neg Q$ Definition of implication
(with P and Q replaced by $\neg P$ and $\neg Q$)

$\Leftrightarrow$ $P \vee \neg Q$ Involution
(substituting $\neg\neg P$ by P)

$\Leftrightarrow$ $\neg Q \vee P$ Commutativity
(with $\neg Q$ replacing Q)

$\Leftrightarrow$ $Q \to P$ Definition of implication
(with P replaced by Q and Q replaced by P)

In this example, I made the use of the substitution and replacement principles explicit. Normally, we just mention the name of the law involved.

……

Example: We prove that $P \rightarrow P \vee Q$ is a tautology; we do that by showing it equivalent to T.

$$\begin{array}{ll}
 & P \rightarrow P \vee Q \\
\Leftrightarrow & \neg P \vee (P \vee Q) \quad \text{Definition of implication} \\
\Leftrightarrow & \underline{(\neg P \vee P)} \vee Q \quad \text{Associativity of OR} \\
\Leftrightarrow & T \vee Q \quad \text{Excluded middle} \\
\Leftrightarrow & T \quad \text{Domination}
\end{array}$$

Example : We prove the distributivity of OR over AND from the distributivity of AND over OR.

Proof. RTP $P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee R)$

Inference

Defn. For two propositional expressions A and B we say that B **can be inferred from** A iff $A \to B$ is a tautology. We say that A is **as weak as** than B and that B is **as strong as** than A. Notation: Either $A \Rightarrow B$ or $B \Leftarrow A$.

Example: $P \Rightarrow P \vee Q$ since $P \to P \vee Q$ is a tautology.

Notes:

- Inference is a refinement of equivalence in that $A \Leftrightarrow B$ exactly if $A \Rightarrow B$ and $B \Rightarrow A$.

We can extend the principle of replacement to inferences:

Principle: **Replacing a logic variable in a tautology.** For any propositional expressions A, B, C and any propositional variable V:

- if B is a tautology then $B[V := A]$ is also a tautology;
- if $B \Leftrightarrow C$ then $B[V := A] \Leftrightarrow C[V := A]$; and
- $\underline{\text{if } B \Rightarrow C \text{ then } B[V := A] \Rightarrow C[V := A].}$

Extending the principle of substitution is a bit trickier.

Principle: **Monotone Substitution.** Let A, B, and C be propositional expressions. Suppose that $A \Rightarrow B$. It is always the case that

- $A \wedge C \Rightarrow B \wedge C$
- $C \wedge A \Rightarrow C \wedge B$
- $A \vee C \Rightarrow B \vee C$
- $C \vee A \Rightarrow C \vee B$
- $C \rightarrow A \Rightarrow C \rightarrow B$

Principle: **Anti-Monotone Substitution.** Let A, B, and C be propositional expressions. Suppose that $A \Rightarrow B$. It is always the case that

- $A \rightarrow C \Leftarrow B \rightarrow C$
- $\neg A \Leftarrow \neg B$

By using monotone and anti-monotone substitution a number of times, we can determine the effect of a substitution involving an isolated part of an expression.

Example: We know that $P \Rightarrow P \vee Q$ so what is the relationship between

$R \wedge \neg P$ and $R \wedge \neg(P \vee Q)$?

- Since $P \Rightarrow P \vee Q$ we have $\neg P \Leftarrow \neg(P \vee Q)$, by anti-monotone substitution.

- Then by monotone substitution we have
$$R \wedge \neg P \Leftarrow R \wedge \neg(P \vee Q)$$

Challenge:

- Develop a definition of algebraic proof, which allows you to prove inferences as well as equivalences.

Duality

Note that many laws of propositional logic come in pairs. E.g.

De Morgan's laws:
$$\begin{aligned}\neg(P \wedge Q) &\Leftrightarrow \neg P \vee \neg Q \\ \neg(P \vee Q) &\Leftrightarrow \neg P \wedge \neg Q\end{aligned}$$

The Principle of Duality: For any law of propositional logic $A \Leftrightarrow B$ involving only propositional variables, AND, OR, NOT, T, and F.

If you replace $\begin{array}{ccc} \wedge & \text{by} & \vee \\ \vee & \text{by} & \wedge \\ \mathrm{T} & \text{by} & \mathrm{F} \\ \mathrm{F} & \text{by} & \mathrm{T} \end{array}$ to get $A' \Leftrightarrow B'$, this too will be a law.

We say that AND and OR are **dual** to each other.

For example: Here is a law about AND, OR and NOT

$$\neg(\neg P \vee Q) \Leftrightarrow P \wedge \neg Q$$

Having proved this law, we immediately get another law

$$\neg(\neg P \wedge Q) \Leftrightarrow P \vee \neg Q$$

by duality.

Why it works: In the truth tables for AND, OR and NOT, we can systematically replace $\begin{array}{ccc} \wedge & \text{by} & \vee \\ \vee & \text{by} & \wedge \\ \text{T} & \text{by} & \text{F} \\ \text{F} & \text{by} & \text{T} \end{array}$ and we still have valid truth tables.

Similarly NAND and NOR are dual to each other, as are BICONDITIONAL and XOR.

E.g. If we know $P \oplus T \Leftrightarrow \neg P$ we also know $P \leftrightarrow F \Leftrightarrow \neg P$

IMPLICATION is dual to an operator $\not\leftarrow$ defined by

$$P \not\leftarrow Q \Leftrightarrow \neg P \wedge Q$$

NOT is dual to itself.

Summary of definitions

- A **statement** is an assertion that may be labelled true or false. **Proposition** is another word for statement.
- A **propositional expression** is an expression made up of
  - the constants T and F
  - any number of propositional variables P, Q, R, ...
  - the propositional operators $\wedge$, $\vee$, $\neg$, ...
  - parentheses
- A propositional expression is a **tautology** iff it evaluates to T regardless of the truth values assigned to its propositional variables.
- A propositional expression is a **contradiction** iff it evaluates to F regardless of the truth values assigned to its propositional variables.
- A propositional expression is a **conditional statement** if it may evaluate to either T or F depending on the values assigned to its propositional variables.
- Propositional expressions A and B are **logically equivalent** iff $A \leftrightarrow B$ is a tautology.

Summary of laws

Commutative operators: $\wedge$, $\vee$, $\leftrightarrow$, $\oplus$.

Associative operators: $\wedge$, $\vee$, $\leftrightarrow$, $\oplus$.

Idempotent operators: $\wedge$, $\vee$

Identities and anti-identity:

$$\begin{aligned} T \wedge P &\Leftrightarrow P \\ F \vee P &\Leftrightarrow P \\ T \rightarrow P &\Leftrightarrow P \\ P \rightarrow F &\Leftrightarrow \neg P \\ T \leftrightarrow P &\Leftrightarrow P \\ F \oplus P &\Leftrightarrow P \end{aligned}$$

Domination:

$$\begin{aligned} F \wedge P &\Leftrightarrow F \\ T \vee P &\Leftrightarrow T \\ F \rightarrow P &\Leftrightarrow T \\ P \rightarrow T &\Leftrightarrow T \end{aligned}$$

Distribution laws:

$$\begin{aligned} P \wedge (Q \vee R) &\Leftrightarrow (P \wedge Q) \vee (P \wedge R) \\ P \vee (Q \wedge R) &\Leftrightarrow (P \vee Q) \wedge (P \vee R) \end{aligned}$$

De Morgan’s laws

$$\neg(P \wedge Q) \;\Leftrightarrow\; \neg P \vee \neg Q$$

$$\neg(P \vee Q) \;\Leftrightarrow\; \neg P \wedge \neg Q$$

Biconditional laws

Definition of biconditional : $(P \leftrightarrow Q) \Leftrightarrow (P \wedge Q) \vee (\neg P \wedge \neg Q)$

Transitivity : $(P \leftrightarrow Q) \wedge (Q \leftrightarrow R) \Rightarrow (P \leftrightarrow R)$

Reflexivity : $P \leftrightarrow P \Leftrightarrow T$

Implication Laws:

Definition of implication : $P \rightarrow Q \Leftrightarrow \neg P \vee Q$

Shunt : $P \rightarrow (Q \rightarrow R) \Leftrightarrow (P \wedge Q) \rightarrow R$

Contrapositive : $P \rightarrow Q \Leftrightarrow \neg Q \rightarrow \neg P$

Modus Ponens : $P \wedge (P \rightarrow Q) \Rightarrow Q$

Transitivity : $(P \rightarrow Q) \wedge (Q \rightarrow R) \Rightarrow (P \rightarrow R)$

Reflexivity : $P \rightarrow P \Leftrightarrow T$

Anti-symmetry : $(P \rightarrow Q) \wedge (Q \rightarrow P) \Leftrightarrow P \leftrightarrow Q$

Other very useful laws:

Involution : $\neg\neg P \Leftrightarrow P$

Definition of XOR : $\neg\,(P \leftrightarrow Q) \Leftrightarrow P \oplus Q$

Contradiction : $\neg P \wedge P \Leftrightarrow F$

Excluded middle : $\neg P \vee P \Leftrightarrow T$

Q & A

Q. I don't understand why there are two symbols: $\leftrightarrow$ and $\Leftrightarrow$. Don't they mean the same thing?

A. The $\leftrightarrow$ symbol is an operator, which combines boolean values T and F according to the rule in it's truth table. The $\Leftrightarrow$ is a relation we use to compare propositional expressions.

If you ask me what the value of $P \leftrightarrow Q$ is, I would say that I don't know because its value depends on what values are assigned to the variables P and Q.

If you asked me whether $P \Leftrightarrow Q$, I can confidently say "no they are not equivalent".

One way to look at it is that the $\leftrightarrow$ symbol is a mathematical operator that combines mathematical values in $\{T, F\}$, just a $+$ is a mathematical operator that combines numerical values. On the other hand $\Leftrightarrow$ is a relation between mathematical expressions, meaning that the two expressions have the same meaning. We might say that $\leftrightarrow$ is part of mathematics, while $\Leftrightarrow$ is a part of meta-mathematics.

Q. Same question for $\rightarrow$ and $\Rightarrow$.

A. Same answer. $\rightarrow$ is an operator, that combines boolean values, whereas $\Rightarrow$ is used to compare propositional expressions.

Q. Then how do $\leftrightarrow$ and $\Leftrightarrow$ relate to good old $=$?

The equals sign is used to combine values of any type to obtain a truth value. For example $1 = 2$ is F. You can think of $\leftrightarrow$ as a version of $=$ which we will use only to combine boolean values. However often people write simply "$A = B$" to mean "$A = B$ is a tautology". For example, someone might write "$2y = y + y$" when clearly what they mean to say is that the "$2y = y + y$ is a tautology". So in this usage the equals sign is being used more like equivalence.

Q. Why not use the same symbols as the digital logic course?

A. (0) As you will see the $\wedge$ and $\vee$ symbols fit nicely with the $\cap$ and $\cup$ symbols used in set theory, which we will see next. (1) The $\wedge$ and $\vee$ symbols are quite common outside of digital logic. (2) The $\wedge$ and $\vee$ symbols are slowly becoming more commonly used in writing about digital logic.