Functions and Relations

Reading 12.1, 12.2, 12.3

Recall Cartesian products and pairs: E.g. $\{1, 2, 3\} \times \{T, F\}$

$$\{(1, T), (1, F), (2, T), (2, F), (3, T), (3, F)\}$$

What is a function?

Informal Defn. A function is a rule that, for each member of one set (the domain), identifies a single member of another set (the range).

Definition: A *binary relation* R consists of 3 things

- a set $\mathrm{dom}(R)$, called its *domain*
- a set $\mathrm{rng}(R)$, called its *range*
- a set $\mathrm{graph}(R)$, called its *graph*. Such that
  - $*$ the graph is set of pairs with the first member from $\mathrm{dom}(R)$ and the second from $\mathrm{rng}(R)$. I.e.

$$\mathrm{graph}(f) \subseteq \mathrm{dom}(R) \times \mathrm{rng}(R)$$

Example: $\mathrm{dom}(R) = \{1, 2, 3, 4\}$ $\mathrm{rng}(R) = \{1, 2, 3, 4\}$

- $\mathrm{graph}(R) = \{(1, 1), (2, 2), (3, 2), (3, 3)\}$

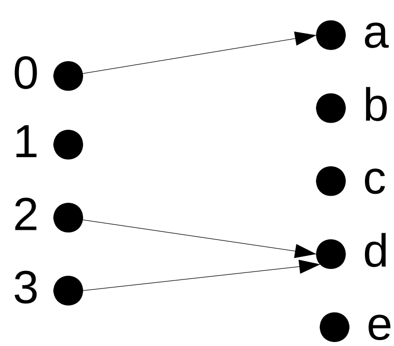
Example: $\mathrm{dom}(R) = \mathrm{rng}(R) = \mathbb{R}$

- $(x, y) \in \mathrm{graph}(R)$ iff $x^2 + y^2 = 1$.

Notation: We write xRy to mean $(x, y) \in \mathrm{graph}(R)$. The text writes $(x, y) \in R$ to mean the same.

Definition: A *partial function* f is a relation such that each member of the domain appears at most once as the first member of a pair in the graph:
$(x, y_0) \in \mathrm{graph}(f) \wedge (x, y_1) \in \mathrm{graph}(f) \to y_0 = y_1$, for all x, y_0, y_1

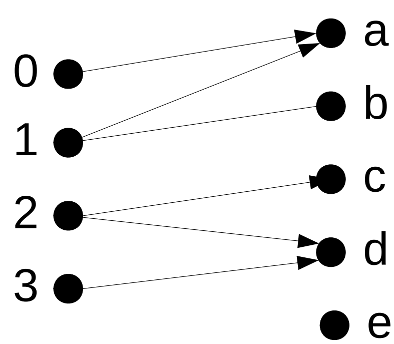
Example: A relation that is a partial function:



Definition: A *total relation* f is a relation such that each member of the domain appears at least once as the first member of a pair in the graph:

$$\forall x \in \mathrm{dom}(f), \exists y \in \mathrm{rng}(f), (x, y) \in \mathrm{graph}(f)$$

Example: A relation that is a total relation:

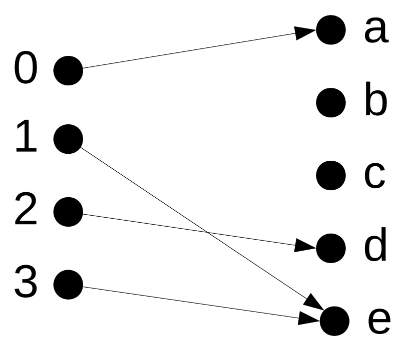


Definition: A *function* f is a relation such that each member of the domain appears at exactly once as the first member of a pair in the graph.

This last requirement can be formalized into two parts

- f is a partial function
- f is a total relation.

Example: A relation that is both a partial function and a total relation:



Note: Every function is a partial function and every partial function is a relation.

Notation:

- We write $f : D \to R$ to mean f is a function with $\mathrm{dom}(f) = D$ and $\mathrm{rng}(f) = R$.
- We write $f : D \rightsquigarrow R$ to mean f is a partial function with $\mathrm{dom}(f) = D$ and $\mathrm{rng}(f) = R$.
- And if f is a partial function or a function, we write $f(x) = y$ to mean $(x, y) \in \mathrm{graph}(f)$

Note: The text does not mention the graph and simply writes $(x, y) \in R$ where I'm writing $(x, y) \in \mathrm{graph}(R)$.

Example: function

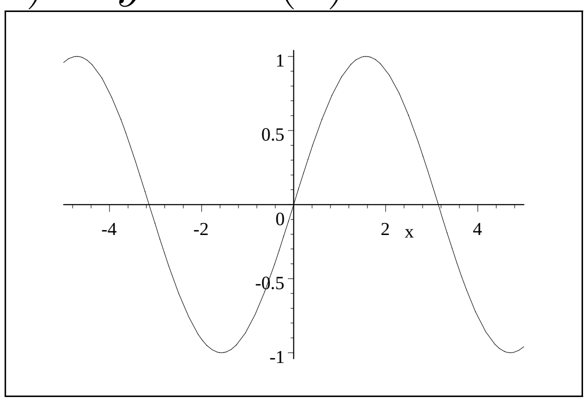
- $f1 : \{0, 1, 2, 3\} \to \{T, F\}$
- $\mathrm{graph}(f1) = \{(0, T), (1, F), (2, T), (3, F)\}$

Example: function

- $f2 : \{0, 1, 2, 3\} \to \{0, 1, ..., 6\}$
- $\text{graph}(f2) = \{(0, 0), (1, 2), (2, 4), (3, 6)\}$

Example: function

- $\sin : \mathbb{R} \to \mathbb{R}$
- $(x, y) \in \text{graph}(\sin)$ iff $y = \sin(x)$



Example: relation

- $\text{dom}(f3) = \{0, 1, 2, 3\}$, $\text{rng}(f3) = \{T, F\}$
- $\text{graph}(f3) = \{(0, T), (1, F), (2, T), (3, F), (0, F)\}$

Example: partial function

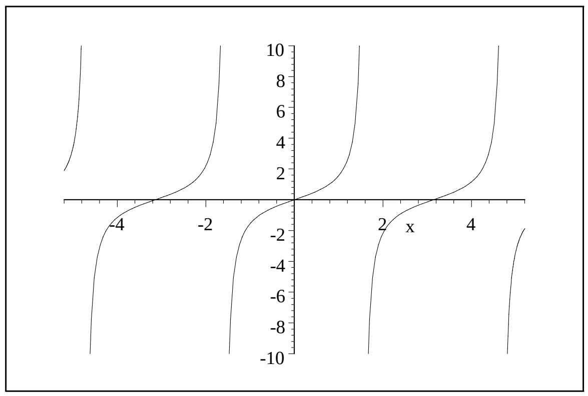
- $f4 : \{0, 1, 2, 3\} \rightsquigarrow \{0, 1, ..., 6\}$
- $\text{graph}(f4) = \{(0, 0), (1, 2), (2, 4)\}$

Example: function

- $f5 : (-\pi/2, \pi/2) \to \mathbb{R}$
- $f5(x) = \tan(x)$

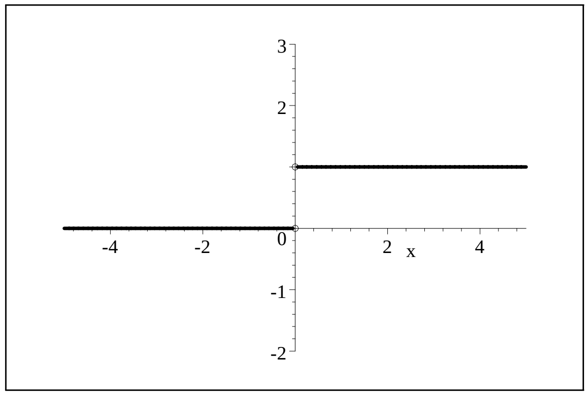
Example: partial function

- $\tan : \mathbb{R} \rightsquigarrow \mathbb{R}$



Example: partial function. The step function.

- $f6 : \mathbb{R} \rightsquigarrow \mathbb{R}$
- $\mathrm{graph}(f6) = \{(x, 0) \mid x \in \mathbb{R} \wedge x < 0\} \cup \{(x, 1) \mid x \in \mathbb{R} \wedge x > 0\}$



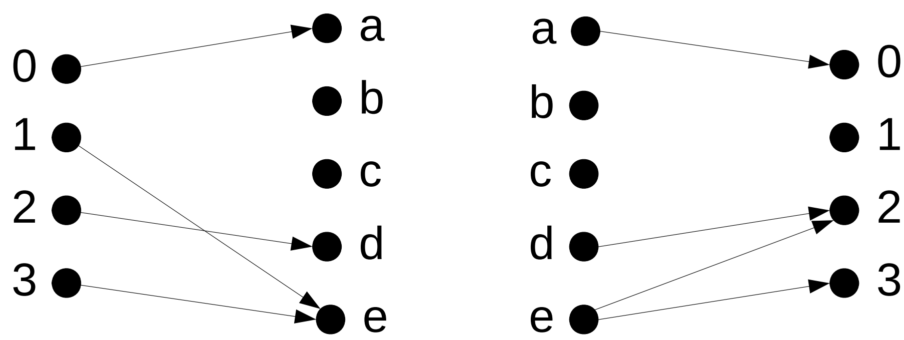
Inversion, one-one, and onto

Definition: The *inverse* of a relation R is a relation R^{-1} such that

- $\mathrm{dom}(R^{-1}) = \mathrm{rng}(R)$
- $\mathrm{rng}(R^{-1}) = \mathrm{dom}(R)$
- $\mathrm{graph}(R^{-1}) = \{(y, x) \mid (x, y) \in \mathrm{graph}(R)\}$

Note that $\left(R^{-1}\right)^{-1} = R$, for all relations R.

Example:



Example: Consider the relation P for parent. xPy if x is y's parent

- Consider $C = P^{-1}$
- Then yCx is true only if x is y's parent
- What is C in English?

Note that the inverse of a function may or may not be a function.

Example: Consider

- $f1 : \{0, 1, 2, 3\} \to \{T, F\}$
- $\mathrm{graph}(f1) = \{(0, T), (1, F), (2, T), (3, F)\}$
- Then $\mathrm{graph}(f1^{-1}) = \{(T, 0), (T, 2), (F, 1), (F, 3)\}$
- This can not be the graph of a function, since T (for example) occurs twice as a the first item of a pair.

Example: Consider

- $f2 : \{0, 1, 2, 3\} \to \{0, 1, ..., 6\}$
- $\mathrm{graph}(f2) = \{(0, 0), (1, 2), (2, 4), (3, 6)\}$
- Then $\mathrm{graph}(f2^{-1})$ is $\{(0, 0), (2, 1), (4, 2), (6, 3)\}$. But the domain of $f2^{-1}$ is $\{0, 1, ..., 6\}$ so the 1 (for example) does not occur as the first member of a pair.
- $f2^{-1}$ is a partial function.

Which relations have inverses that are functions?

Definition: A relation is *one-one* if every member of the range appears at most once as the second member of some pair in the graph.

Theorem:

- The inverse of a one-one relation is a partial function.
- The inverse of a partial function is a one-one relation.

Definition: A relation is *onto* if every member of the range appears at least once as the second member of some pair in the graph.

Theorem:

- The inverse of an onto relation is a total relation.

- The inverse of a total relation is an onto relation.

Theorem:

- The inverse of a one-one and onto relation is a function.
- And the inverse of a function is a one-one and onto relation.

Corollary: The inverse of a one-one and onto function is a one-one and onto function.

Example:

- $f7 : \mathbb{Z} \to \mathbb{Z}$, $\mathrm{graph}(f7) = \{(n, n+10) \mid n \in \mathbb{Z}\}$
- This function is one-one and onto.
- Its inverse is a function $f7^{-1} : \mathbb{Z} \to \mathbb{Z}$, $\mathrm{graph}(f7^{-1}) = \{(n, n-10) \mid n \in \mathbb{Z}\}$

Example: Consider a function from 16 bit strings to 16 bit strings which swaps the first and second byte of the string

- $swap : \{F, T\}^{16} \to \{F, T\}^{16}$

- $$swap(\langle b_{15}, b_{14}, b_{13}, b_{12}, b_{11}, b_{10}, b_9, b_8, b_7, b_6, b_5, b_4, b_3, b_2, b_1, b_0\rangle)$$
  $$= \langle b_7, b_6, b_5, b_4, b_3, b_2, b_1, b_0, b_{15}, b_{14}, b_{13}, b_{12}, b_{11}, b_{10}, b_9, b_8\rangle$$
- This one-one onto function is its own inverse. $swap^{-1} = swap$.

Identity and composition

Identity function. For each set A, the function $id_A : A \to A$ maps each element or A to itself.

$$id_A(x) = x\text{, for all } x \in A$$

Composition.

Consider the relation P for parent. xPy iff x is y's parent

- Define a relation Q so that xQy iff there is a z such that zPx and zPy.
- What is Q in English?

Consider the relation xQy meaning x is y's sibling

- Define relation K so that xKy iff there is are w and z such that wPy and wQz and zPx.
- What is K in English?

Defn: Suppose $\mathrm{rng}(R) = \mathrm{dom}(S)$. The *composition of* S *following* R, written $S \circ R$ is a relation such that

- $\mathrm{dom}(S \circ R) = \mathrm{dom}(R)$
- $\mathrm{rng}(S \circ R) = \mathrm{rng}(S)$
- $\mathrm{graph}(S \circ R)$ is such that

$(x\,(S \circ R)\,y$ iff $\exists z, xRz \wedge zSy)$, for all $x \in \mathrm{dom}(R), y \in \mathrm{rng}(S)$

Example: $Q = P \circ P^{-1}$

Example: $K = P \circ Q \circ P^{-1}$

Example: Suppose that f and g are functions, then

$$(f \circ g)(x) = f(g(x)), \text{ for all } x \in \mathrm{dom}(g)$$

Note that $\circ$ is associative and has identity id and the empty relation is a dominator.

$$\begin{aligned} T \circ (S \circ R) &= (T \circ S) \circ R \\ R \circ id &= R = id \circ R \\ R \circ \emptyset &= \emptyset = \emptyset \circ R \end{aligned}$$

In general $\circ$ is not commutative, nor is it idempotent.

$$S \circ R \text{ may not equal } R \circ S$$

$$R \circ R \text{ may not equal } R$$

Suppose that a relation R has $\mathrm{dom}(R) = \mathrm{rng}(R) = A$.

- Then R^0 is id_A
- $R^1 = R$
- $R^2 = R \circ R$
- $R^3 = R \circ R \circ R$
- Etc.

Example: Suppose that xRy means that two nodes in a network are directly connected (1 hop)

- Then $x(R \circ R)y$ means that x and y are connected by 2 hops.
- and $id \cup R \cup (R \circ R)$ means[1] that 2 nodes are connected by 0, 1, or 2 hops.
- Define $R^0 = id$, $R^1 = R$, $R^n = (R \circ R^{n-1})$ for $n \geq 1$
- Then $R^0 \cup R^1 \cup R^2 \cup \cdots$ is a relation that indicates whether two computers are connected by any number of hops.
- This is called the reflexive and transitive closure of R.
- The notation is R^*

[1] The union of relations is the relation formed by unioning the domains, ranges, and graphs.

We can compute the reflexive and transitive closure of R as follows

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$T := id_A$; // Where $\mathrm{dom}(R) = \mathrm{rng}(R) = A$
$U := id_A$
$i := 0$;
// Invariant: $T = \bigcup_{j \in \{0,1,\ldots,i\}} R^j$ and $U = R^i$
while(true) {
    $U := U \circ R$;
    if($U \subseteq T$) break ;
    $T := U \cup T$;
    $i := i + 1$ }

This is very useful, for example, to determine if a network is fully connected.

Relational Databases

Currently most database management systems are based on the “relational model”.

Examples include, Access, Oracle, and MySQL.

Tables and Databases

A *table* (or n-ary relation) R has

- A tuple of n distinct attribute names $\mathrm{attr}(R) = (c_0, c_1, ...c_{n-1})$
- n domain sets $\mathrm{dom}(R) = (D_0, D_1, \cdots, D_{n-1})$
- $\mathrm{graph}(R) \subseteq D_0 \times D_1 \times \cdots \times D_{n-1}$

We can visualize a table as a matrix in which

- each column has a name and is associated with a set of potential values
- no row is repeated
- the order of the rows does not matter

Examples:

Personnel

personnel-num	**name**	**salary**	**boss**
001	Sue King	100000	001
002	Fong Ping	40000	001
999	Bob Will	20000	001

Projects:

Name	**Assigned**	**Completion-date**
Snipe	001	2003-12-31
Snipe	999	2003-12-31
Snark	999	2004-01-31

A *relational database* is

- a set of m table names $\{t_0, t_1, ..., t_{m-1}\}$
- m tables indexed by name $T_{t_0}, T_{t_1}, ..., T_{t_{m-1}}$

Example: The set of table names is $\{personnel, projects\}$ and the tables $T_{personnel}$ and $T_{projects}$ are the tables above.

Query operations on data bases

Query operations: projection, attribute renaming, selection, join.

Projection:

- Given a tuple $p = (v_0, v_1, \cdots v_{n-1})$ from a table T with attributes $(c_0, c_1, ...c_{n-1})$. Consider a sequence of distinct attributes $a' = (c_{i_0}, c_{i_1}, ..., c_{i_{k-1}})$
  - define the *projection* of p onto a' (written $p[(c_{i_0}, c_{i_1}, ..., c_{i_{k-1}})]$) to be the tuple $(v_{i_0}, v_{i_1}, ..., v_{i_{k-1}})$
- For a table T define the *projection* of T onto a' as a table T' with
  - attributes a'
  - domains $(D_{i_0}, D_{i_1}, \cdots, D_{i_{k-1}})$
  - graph

$$\{p[(c_{i_0}, c_{i_1}, ..., c_{i_{k-1}})] \mid p \in \mathrm{graph}(R)\}$$

Example: If we want to know who works for whom, but hide salary information, we can project out the salary:

- Personnel[personnel-num, name, boss]

Suppose we want to know who has a management position:

- Personnel[boss] gives

boss
001

Attribute Renaming.

- Sometimes we need to rename the attributes. We can combine this with projection. E.g.
- Projects[name $\leadsto$ project-name, assigned $\leadsto$ personnel-num]
- This is the same table as Projects[name, assigned], except with different attribute names.

Selection:

- Suppose T is a table with attributes $(c_0, c_1, ...c_{n-1})$ and E is a boolean expression with variable names drawn from $\{c_0, c_1, ...c_{n-1}\}$. Then
$$T \mid E$$
is a table with attributes and domains the same as T and graph
$$\{(c_0, c_1, ...c_{n-1}) \in \mathrm{graph}(T) \mid E\}$$
- Example: suppose we want to know all the personnel making more than 50000
$$\text{Personnel} \mid \text{salary} > 50000$$
- Example: Bob wants to know the names of all his

projects due this year

(projects | assigned=999 $\wedge$ completion-date $<$ 2004-01-01)
[name]

Join:

- Join combines two tables.
- Consider tables
  - Names

student-num	name
12345	Smith
23456	Jones
11235	Seth
31415	Lee

and

  - Marks

student-num	mark
12345	A+
23456	B
11235	B+
31415	F

- Then the join Names*Marks is

- Suppose A and B are tables with attribute names
$$\text{attr}(A) = (a_0, a_1, ...a_{m-1})$$
$$\text{attr}(B) = (b_0, b_1, ..., b_{n-1})$$
and domains
$$\text{dom}(A) \ = \ (A_0, A_1, ..., A_{m-1})$$
$$\text{dom}(B) \ = \ (B_0, B_1, ..., B_{n-1})$$
- We say A and B are *join-compatible* iff equally named attributes correspond to equal domains. I.e. iff $a_j = b_k$ implies $A_j = B_k$ (for all j, k)
- The *join* of join-compatible tables A and B, $A * B$, is a table C such that
  - $*$ the set of attributes is the union of the sets of attributes of A and B
    i.e. if
$$\text{attr}(C) = (c_0, c_1, ...c_{p-1})$$

then

$$\{c_0, c_1, ..., c_{p-1}\} = \{a_0, ...a_{n-1}\} \cup \{b_0, ..., b_{m-1}\}$$

- $*$ the domains correspond to the domains in A and B. I.e. if

$$\mathrm{dom}(C) = (C_0, ...C_{p-1})$$

then (for all i, j, k) if $c_i = a_j$ then $C_i = A_j$ and if $c_i = b_k$ then $C_i = B_k$.

- $*$ The graph consists of tuples that combine the values from tuples in A and B.
- $*$ I.e. x is a tuple of C iff there exist tuples y from A and z from B such that

$$x[\mathrm{attr}(A)] = y$$

and

$$x[\mathrm{attr}(B)] = z$$

- $*$ Note that y and z must agree on the values of any common attributes.

- Example: I want to know the names of people assigned to various projects

Projects[name $\leadsto$ project-name,assigned $\leadsto$ personnel-num]
* Personnel[personnel-num,name]

- Gives

- How do we make this table?

personnel-num	**name**	**boss**	**boss-name**
001	Sue King	001	Sue King
002	Fong Ping	001	Sue King
999	Bob Willing	001	Sue King

Note that if we have binary relations then composition is essentially a join followed by a projection. I.e. if we regard a binary relation as a table having attributes *left* and *right*.

$S \circ R$ is $(S[\textit{left} \rightsquigarrow \textit{middle,right}] * R[\textit{left}, \textit{right} \rightsquigarrow \textit{middle}])[\textit{left}, \textit{right}]$

SQL

- SQL is the standard (and most popular) data-base query language. It is based (loosely) on the query operations presented above.