Pascal's triangle and the Fibonacci sequence:

Consider Pascal's triangle

$n \backslash r$	0	1	2	3	4	5	_	$n \backslash r$	0	1	2	3	4	5
0	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$							0	1					
1	$\binom{1}{0}$	$\binom{1}{1}$						1	1	1				
2	$\binom{2}{0}$	$\binom{1}{2}$	$\binom{2}{2}$				=	2						
2 3		$\binom{3}{1}$	$\binom{3}{2}$	$\binom{3}{2}$				3	1	3	3	1		
4	(4)	$\binom{4}{1}$	$\binom{4}{2}$	$\binom{4}{2}$	$\binom{4}{4}$			4	1	4	6	4	1	
5	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$\binom{1}{5}$	$\begin{pmatrix} 2 \\ 2 \end{pmatrix}$ $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ $\begin{pmatrix} 4 \\ 2 \end{pmatrix}$ $\begin{pmatrix} 5 \\ 2 \end{pmatrix}$	$\begin{pmatrix} 3 \\ 3 \end{pmatrix}$ $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$ $\begin{pmatrix} 5 \\ 3 \end{pmatrix}$	$\binom{4}{5}$	$\binom{5}{5}$		5	1	5	10	10	5	1

which is defined by Pascal's laws:

$$\begin{array}{lll} \binom{n}{n} & = & 1 & \quad \text{, for all } n \in \mathbf{N} \\ \binom{n}{0} & = & 1 & \quad \text{, for all } n \in \mathbf{N} \\ \binom{n}{r} & = & \binom{n-1}{r-1} + \binom{n-1}{r} & \text{, for all } n, r \in \mathbf{N}, \text{ such that } 0 < r < n \\ \end{array}$$

Adding up the South-East to North-East diagonals reveals an interesting pattern:

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = 1 = 1 = \text{fib}(0)$$

$$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 1 + 1 = 2 = \text{fib}(1)$$

$$\begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 1 + 2 = 3 = \text{fib}(2)$$

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = 1 + 3 + 1 = 5 = \text{fib}(4)$$

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} = 1 + 4 + 3 = 8 = \text{fib}(5)$$

where fib is the Fibonacci sequence defined by

$$fib(0) = 1 \tag{1}$$

$$fib(1) = 1 \tag{2}$$

$$fib(n) = fib(n-2) + fib(n-1)$$
, for all $n \in \{2, 3, ...\}$

This leads to a conjecture

$$\operatorname{fib}(n) = \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n-i}{i}$$
 , for all $n \in \mathbb{N}$ (4)

(remember that $\lfloor x \rfloor$ is the integer part of a real number x.) Prove this conjecture using complete induction.