Pascal's triangle and the Fibonacci sequence:

Consider Pascal's triangle

$$
\begin{array}{c|cccccc}
n\backslash r & 0 & 1 & 2 & 3 & 4 & 5 \\
\hline
0 & \binom{0}{0} \\
1 & \binom{1}{0} & \binom{1}{1} \\
2 & \binom{2}{0} & \binom{2}{1} & \binom{2}{2} \\
3 & \binom{3}{0} & \binom{3}{1} & \binom{3}{2} & \binom{3}{3} \\
4 & \binom{4}{0} & \binom{4}{1} & \binom{4}{2} & \binom{4}{3} & \binom{4}{4} \\
5 & \binom{5}{0} & \binom{5}{1} & \binom{5}{2} & \binom{5}{3} & \binom{5}{4} & \binom{5}{5}
\end{array}
=
\begin{array}{c|cccccc}
n\backslash r & 0 & 1 & 2 & 3 & 4 & 5 \\
\hline
0 & 1 \\
1 & 1 & 1 \\
2 & 1 & 2 & 1 \\
3 & 1 & 3 & 3 & 1 \\
4 & 1 & 4 & 6 & 4 & 1 \\
5 & 1 & 5 & 10 & 10 & 5 & 1
\end{array}
$$

which is defined by Pascal's laws:

$$
\begin{array}{lcll}
\binom{n}{n} & = & 1 & \text{, for all } n \in \mathbf{N} \\
\binom{n}{0} & = & 1 & \text{, for all } n \in \mathbf{N} \\
\binom{n}{r} & = & \binom{n-1}{r-1} + \binom{n-1}{r} & \text{, for all } n, r \in \mathbf{N}\text{, such that } 0 < r < n
\end{array}
$$

Adding up the South-East to North-East diagonals reveals an interesting pattern:

$$
\begin{array}{lclclcl}
\binom{0}{0} & = & 1 & = & 1 & = & \mathrm{fib}(0) \\
\binom{1}{0} & = & 1 & = & 1 & = & \mathrm{fib}(1) \\
\binom{2}{0} + \binom{1}{1} & = & 1+1 & = & 2 & = & \mathrm{fib}(2) \\
\binom{3}{0} + \binom{2}{1} & = & 1+2 & = & 3 & = & \mathrm{fib}(3) \\
\binom{4}{0} + \binom{3}{1} + \binom{2}{2} & = & 1+3+1 & = & 5 & = & \mathrm{fib}(4) \\
\binom{5}{0} + \binom{4}{1} + \binom{3}{2} & = & 1+4+3 & = & 8 & = & \mathrm{fib}(5)
\end{array}
$$

where fib is the Fibonacci sequence defined by

$$\mathrm{fib}(0) = 1 \tag{1}$$

$$\mathrm{fib}(1) = 1 \tag{2}$$

$$\mathrm{fib}(n) = \mathrm{fib}(n-2) + \mathrm{fib}(n-1) \quad \text{, for all } n \in \{2, 3, ...\} \tag{3}$$

This leads to a conjecture

$$\mathrm{fib}(n) = \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n-i}{i} \quad \text{, for all } n \in \mathbb{N} \tag{4}$$

(remember that $\lfloor x \rfloor$ is the integer part of a real number x.)

Prove this conjecture using complete induction.