Problem set 7

Engineering 3422, 2005

Due Nov 1, 2005

Q0

Given sets S and T we can define the set of labelled binary trees over set S and T by the following two rules.

- For any t in T, $\langle t \rangle$ is a labelled binary tree of height 0. Such trees are called leaves.
- For each element s in S, the triple $\langle x, s, y \rangle$ is a labelled binary tree of height $n+1$ if x and y are labelled binary trees of heights less than $n+1$ and at least one of x and y is has height n. Such trees are called nodes.

For example $\langle \langle \langle p \rangle, a, \langle q \rangle \rangle, b, \langle \langle r \rangle, c, \langle \langle s \rangle, d, \langle t \rangle \rangle \rangle \rangle$ is a labelled binary tree of height 3 over the sets $\{a, b, c, d\}$ and $\{p, q, r, s, t\}$.

We can define a leaf counting function by:

$$\begin{aligned} lc(\langle t \rangle) &= 1 \\ lc(\langle x, s, y \rangle) &= lc(x) + lc(y) \end{aligned}$$

We can define a node counting function by

$$\begin{aligned} nc(\langle t \rangle) &= 0 \\ nc(\langle x, s, y \rangle) &= 1 + nc(x) + nc(y) \end{aligned}$$

Show by complete induction that for any labelled binary tree A, $lc(A) = 1 + nc(A)$.

Q1

- I claim that if we define

$$\begin{aligned} \text{minsize}(0) &= 0 \\ \text{minsize}(1) &= 1 \\ \text{minsize}(h) &= \text{minsize}(h-1) + \text{minsize}(h-2) + 1, \text{ for } h \geq 2 \end{aligned}$$

then

$$\text{minsize}(h) = \text{fib}(h+1) - 1$$

where

$$\begin{aligned}
\text{fib}(0) &= 1 \\
\text{fib}(1) &= 1 \\
\text{fib}(n) &= \text{fib}(n-1) + \text{fib}(n-2), \text{ for } n \geq 2
\end{aligned}$$

Prove this claim. [Hint: use complete induction with two base cases.]

[As always with induction problem: State clearly the what property is being proved. State what must be proved in the base step(s). Prove the base step(s). State what must be proved in the inductive step. Prove the inductive step, stating clearly what the inductive hypothesis is and where you make use of it.]