Chapter 8
Optimization Principles

8.1 Introduction

Optimization is the process of finding conditions which give maximum or minimum values of a particular function. It is important to understand that optimization is done within the confines of a concept. We address trade-offs or give and take behaviour in a system to determine the best operating point. In some cases, even simple objectives entail complicated analysis. We will focus on some simple methods which can be applied to a host of engineering problems with limited use of mathematical tools such as software.

8.1.1 The Optimization Problem

In the mathematical statement of optimization, there is a single function:

\[ y = f(x_1, x_2, x_3, x_4, \ldots x_n) \rightarrow (\text{min/max}) \]  

(8.1)

which is referred to as the objective function and the \( x_i \) are the independent variables. The objective function may also be subject to constraints of the form:

\[ g_i(x_1, x_2, x_3, x_4, \ldots x_n) = 0 \quad i = 1 \ldots n \]

(8.2)

where \( g_i \) represent equality constraints and \( h_j \) represent inequality constraints. A given problem may be unconstrained or constrained. We will examine both types of problems and methods to solve them. If a problem only contains linear equations, then it is often denoted as a linear programming problem (LP), while if one or more equations in the problem statement are non-linear, then it is referred to as a non-linear programming problem (NLP).

The field of optimization is very broad. We will examine only select methods which utilize calculus methods using analytical, approximate analytical, and numerical methods.
8.1.2 Basic Steps in an Optimization Problem

The following steps summarize the process of undertaking an optimization:

- Analyze the process or system such that all of the specific characteristics and variables are defined.
- Determine the criterion for optimization and specify the objective function in terms of the above variables.
- Develop a mathematical model for the process or system that relates the input/output variables using well known physical principles (mass, energy, momentum etc.) and empirical relationships.
- If the problem is too large in scope, then break it up into smaller parts, or simplify the objective and the model.
- Apply a suitable optimization technique.
- Check answers and examine sensitivity to changes in coefficients and assumptions.

Example 8.1

Consider the design of a large steel storage tank of diameter \( D \) and height \( H \). The goal is to maximize the volume \( V \) or capacity of the tank for a fixed amount of material (or surface area, \( A \)).

Solution

The mathematical statement of the objective function is:

\[
V = \frac{\pi D^2 H}{4} \rightarrow \text{maximize} \tag{8.3}
\]

subject to

\[
A = \frac{\pi D^2}{2} + \pi DH \tag{8.4}
\]

It should be stated that this is a relatively trivial example, since we have not really said how the material of area \( A \) is being utilized. For example, if the material was a large sheet, how are the ends and sides to be cut from this sheet?

We may combine the two equations to solve the problem as a simpler unconstrained problem by writing the height \( H \) of the tank as:

\[
H = \frac{\left[ A - \frac{\pi D^2}{2} \right]}{\pi D} \tag{8.5}
\]

This gives the following objective function:
\[ V = \frac{AD}{4} - \frac{\pi D^3}{8} \]  

(8.6)

We can now find the optimal diameter by taking the derivative of the objective function, equating to zero, and solving:

\[ \frac{dV}{dD} = \frac{A}{4} - \frac{3\pi D^2}{8} = 0 \]  

(8.7)

which gives:

\[ D_{\text{opt}} = \sqrt{\frac{2A}{3\pi}} \]  

(8.8)

and

\[ H_{\text{opt}} = \sqrt{\frac{2A}{3\pi}} \]  

(8.9)

The maximum volume of the tank can be found to be:

\[ V_{\text{max}} = \pi \left( \frac{2A}{3\pi} \right)^{3/2} \]  

(8.10)

The aspect ratio of the tank is \( H/D = 1 \). Additional constraints requiring the aspect ratio to lie within a particular range could also have been specified.

We have just seen from Example 8.1, how to formulate an optimization problem. In the optimization of engineering systems, many different objectives arise. These include: minimize first cost of a system, minimize life time costs, maximize profits, minimize or maximize heat transfer, minimize pumping power, minimize weight, minimize or maximize work transfer, minimize surface area, maximize volume, minimize pressure drop, minimize entropy generation, etc.

**Example 8.2**

Re-examine the tank problem in Example 8.1. This time consider the volume to be fixed. We desire to obtain the system which has the least surface area \( A \). This could be for weight purposes or for heat transfer reduction. Formulate the objective function for minimum surface area, minimum weight, and minimum cost.

### 8.2 Method of Intersecting Asymptotes

The method of intersecting asymptotes is a simple solution methodology for obtaining approximate solutions to optimization problems. It is based on the principle that two competing trade off behaviours have an optimal solution close to the intersection point of these competing behaviours. This approach which was developed by Bejan et al. (1996), allows complex systems to be optimized for which exact system behaviour may be known or unknown.
For example, often in heat transfer and fluid flow process, the competing behaviour of short and long flow passage length arises, particularly in heat exchanger systems. More often than not thermal and fluid behaviour is correlated using functions of the form:

\[ y = [y_0^n + y_\infty^n]^{1/n} \]  

or

\[ y = \left[ \frac{1}{y_0^n} + \frac{1}{y_\infty^n} \right]^{-1/n} \]

where \( y_0 \) and \( y_\infty \) represent limiting behaviour of the system.

The use of functions in the form of the above equations usually complicates problems and makes them more difficult to solve. Further complicating matters, is the fact that often the correlating parameter \( n \), is not known for many geometries, unless one constructs a new model of this form. In the current approach we do not require the correct form of the heat transfer or fluid flow model, but merely the limiting behaviour, i.e \( y_0 \) and \( y_\infty \).

**Example 8.3**

Consider a particular set of characteristics of an engineering system. It has been found that for small values of the independent variable \( x \), the system behaves according to \( y_0 = 1/x \). While for large values of the independent variable, the system behaves according to \( y_\infty = x^2 \). Determine the exact and approximate values of \( x_{opt} \) which optimizes the system behaviour, i.e. min/max, if the system behaviour is modelled as:

\[ y = y_0 + y_\infty \rightarrow minimize \]  

and

\[ y = \left[ \frac{1}{y_0} + \frac{1}{y_\infty} \right]^{-1} \rightarrow maximize \]

**Example 8.4**

Consider a finite volume \( V = HWL \) which contains several parallel plate channels. Using the method of intersecting asymptotes, determine the optimal plate spacing \( b_{opt} \), such that the heat transfer rate per unit volume is maximum.

**Solution**

We may derive a general result applicable to any duct shape. However, in the case of an array of closely spaced channels, \( b \rightarrow 0 \), the enthalpy balance for fully developed flow gives:

\[ Q_s = \rho \bar{U} N A C_p (\bar{T}_s - T_i) \]
where \( A = bH \) is the cross-sectional area of an elemental duct or channel, \( N = W/b \) is the total number of ducts or channels, \( \bar{T}_s \) is the mean wall temperature, and \( T_i \) is the fluid inlet temperature.

The mean velocity, \( \bar{U} \), in any one duct or channel assuming uniform flow distribution, may be determined from the fully developed flow friction factor - Reynolds number product defined as:

\[
f Re_{Dh} = \frac{2A/P(\Delta p/L)D_h}{\mu U} \quad (8.16)
\]

or

\[
\bar{U} = \frac{2A\Delta pD_h}{\mu PL(Re_{Dh})} \quad (8.17)
\]

Combining the above two results gives the heat transfer rate in terms of the fundamental flow quantities:

\[
Q_s = \frac{2\rho C_p A^2 \Delta p D_h N(\bar{T}_s - T_i)}{\mu PL(Re_{Dh})} \quad (8.18)
\]

In the case of parallel plates \( f Re_{Dh} = 24 \) for laminar flow and \( D_h = 2b \), where \( b \) is the plate spacing.

In the case of an array of channels with large plate spacing, the heat transfer rate may be adequately approximated as boundary layer flow in this limit. The heat transfer rate is determined from:

\[
Q_l = \bar{h}N PL(\bar{T}_s - T_i) \quad (8.19)
\]

where \( \bar{h} \) may be defined from the expression for laminar boundary layer flow over a flat plate:

\[
\frac{\bar{h}L}{k_f} = 0.664 \left( \frac{U_{\infty} L}{\nu} \right)^{1/2} Pr^{1/3} \quad (8.20)
\]

The free stream velocity \( U_{\infty} \), is obtained from a force balance on the array:

\[
\tau_w PN = NA \Delta p \quad (8.21)
\]

where the mean wall shear stress is obtained from the boundary layer solution:

\[
\frac{1}{2} \rho U_{\infty}^2 = 1.328 \left( \frac{U_{\infty} L}{\nu} \right)^{-1/2} \quad (8.22)
\]

Combining Eqs. (8.21) and (8.22) gives the following result for \( U_{\infty} \):

\[
U_{\infty} = 1.314 \left( \frac{\Delta p A}{PL^{1/2} \rho^{1/2} \nu^{1/2}} \right)^{2/3} \quad (8.23)
\]
Finally, combining Eqs. (8.20) and (8.23) yields the following result for the heat transfer rate:

\[
Q_I = 0.7611Nk_f(T_s - T_i) \left( \frac{\Delta p A P^2 L P_r}{\rho \nu^2} \right)^{1/3}
\]

(8.24)

The optimal duct or channel size may be found by means of the method of intersecting asymptotes. The exact shape of the heat transfer rate curve may be found using more exact methods. However, the intersection point of the two asymptotic results is relatively close to the exact point. In this way, an approximate value for the reference duct dimension may be found. Intersecting Eqs. (8.18) and (8.24) gives:

\[
0.7611 \frac{k_f(T_s - T_i)}{\rho^{1/3} \nu^{2/3}} \frac{\Delta p^{1/3} P_r^{1/3}}{N} \left( A P^2 L \right)^{1/3} \approx 2 \frac{\rho C_p(T_s - T_i) \Delta p}{\mu} \frac{N A^2 D_h}{P L (f Re_{D_h})}
\]

(8.25)

\[
\text{Fig. 1 - Method of Intersecting Asymptotes}
\]

After simplifying and collecting the system and geometry terms, the above equation may be written in the following form:

\[
Be^{1/4} \approx \frac{1.656 L (f Re_{D_h})^{3/8}}{D_h}
\]

(8.26)

where \(Be = \Delta p L^2 / \mu \alpha\) is referred to as the Bejan number and \(D_h = 4A/P\) is the hydraulic diameter. The right hand side is only a function of the duct shape and aspect ratio, while the left hand side is a system parameter which is constant and independent of duct shape or aspect ratio once a cooling volume, \(V = HWL\), is specified.
This result can now be applied to an array of parallel plate channels or to arrays of other shapes that are often used in convection cooling of finite volumes. Finally, all of the results in the present analysis are applicable for laminar flows or in terms of the Bejan number, the range defined by:

\[ Be^{1/4} \lesssim 10^3 Pr^{1/2} \quad (8.27) \]

The maximum heat transfer rate for a fixed volume can be obtained from Eq. (8.18) using the optimal result determined by Eq. (8.24). The number of ducts or channels \( N \), which appears in the final result may then be cast in terms of the cooling volume cross-section. In this way, the maximum heat transfer per unit volume may be determined. Subsequent results may then be presented in terms of the following dimensionless heat transfer per unit volume:

\[ Q^* \lesssim \frac{QL^2}{k(T_s - T_i)} = C Be^{1/2} \quad (8.28) \]

where \( Q = Q/(HWL) \) is the heat transfer per unit volume, and \( C \) is a numerical constant determined from the duct geometry.

The problem of a finite volume cooled by a stack of parallel plates (or channels) was first considered by Bejan and Sciubba (1991), see Bejan et al. (1996). The following geometric parameters are required in the general analysis:

\[ D_h = 2b \quad (8.29) \]
\[ f Re_{D_h} = 24 \quad (8.30) \]

Substituting the above results into Eq. (8.26) gives the following results for the optimal plate spacing:

\[ \frac{b_{opt}}{L} \approx 2.726 Be^{-1/4} \quad (8.31) \]

The maximum heat transfer rate for \( N \sim W/b \) channels is:

\[ Q^* \lesssim 0.6192 Be^{1/2} \quad (8.32) \]

The exact values of the constants in Eqs. (8.31,8.32) determined through more rigorous analysis are: 3.025-3.078 and 0.479-0.527, depending on the fluid Prandtl number, \( 0.72 < Pr < 1000 \).

### 8.3 Newton-Raphson Method in Optimization

The Newton-Raphson method for a single variable was introduced in the last chapter. The method is useful in optimization problems provided the objective function is twice differentiable.
Given an objective function \( f(x) \) and solution function:

\[
g(x) = \frac{df}{dx} = 0
\]  

(8.33)

the solution for a variable \( x \) is found by making an initial guess \( x_i \) and solving the following equation:

\[
x_{i+1} = x_i - \frac{g(x_i)}{g'(x_i)} = x_i - \frac{f'(x_i)}{f''(x_i)}
\]  

(8.34)

for the improved solution \( x_{i+1} \). The procedure is repeated until desired convergence is achieved, which usually occurs in fewer than three iterations.

If our objective function is multivariable, the Newton-Raphson method can also be applied. Given an objective function \( f(x_1, x_2, x_3, x_4, \ldots, x_n) \), we must solve the system:

\[
g_1(x_1, x_2, x_3, x_4, \ldots, x_n) = \frac{df}{dx_1} = 0
\]

\[
g_2(x_1, x_2, x_3, x_4, \ldots, x_n) = \frac{df}{dx_2} = 0
\]

\[
g_3(x_1, x_2, x_3, x_4, \ldots, x_n) = \frac{df}{dx_3} = 0
\]

\[\vdots\]

\[
g_n(x_1, x_2, x_3, x_4, \ldots, x_n) = \frac{df}{dx_n} = 0
\]  

(8.35)

To solve the above system we may write the general non-linear system in a linearized form using a truncated Taylor series expansion of each equation:

\[
\begin{bmatrix}
\frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \cdots & \frac{\partial g_1}{\partial x_n}\\
\frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} & \cdots & \frac{\partial g_2}{\partial x_n}
\end{bmatrix}
\begin{bmatrix}
\Delta x_1 \\
\Delta x_2 \\
\vdots \\
\Delta x_n
\end{bmatrix}
= 
\begin{bmatrix}
g_1 \\
g_2 \\
\vdots \\
g_n
\end{bmatrix}_{x_0}
\]  

(8.36)
or in terms of the original objective function:

\[
\begin{bmatrix}
\frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\
\frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2}
\end{bmatrix}
\begin{bmatrix}
\Delta x_1 \\
\Delta x_2 \\
\vdots \\
\Delta x_n
\end{bmatrix}
= \begin{bmatrix}
\frac{\partial f}{\partial x_1} \\
\frac{\partial f}{\partial x_2} \\
\vdots \\
\frac{\partial f}{\partial x_n}
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n
\end{bmatrix}
\tag{8.37}
\]

We may now solve the above matrix as before, repeating iterations as required until desired convergence of the solution is achieved. The matrix of second order partial derivatives is also referred to as the Hessian matrix. It will be utilized in the next section to determine the optimality of a solution, i.e. minimum, maximum, or saddle point.

### 8.4 Tests for Optimality

Up to this point we have not yet determined whether an obtained solution is indeed maximum or minimum. In order to check whether or not a given solution is maximum or minimum, we must return to the calculus of optimization.

#### 8.4.1 Single Variable Functions

Given a single variable function \( f(x) \), whose optimal solution \( x = x_{\text{opt}} \) satisfies:

\[
\frac{df}{dx} \bigg|_{x=x_{\text{opt}}} = 0 \tag{8.38}
\]

the test for optimality requires that we examine the value of the second derivative of \( f(x) \), such that:

\[
\frac{d^2 f}{dx^2} \bigg|_{x=x_{\text{opt}}} \geq 0 \quad \text{min} \tag{8.39}
\]

\[
\frac{d^2 f}{dx^2} \bigg|_{x=x_{\text{opt}}} \leq 0 \quad \text{max} \tag{8.40}
\]

Further, if the second derivative is equal to zero:

\[
\frac{d^2 f}{dx^2} \bigg|_{x=x_{\text{opt}}} = 0 \tag{8.41}
\]
then we must consider further tests. Given a solution $x_{opt}$ which is referred to as a stationary point, that is it satisfies the condition that $f'(x_{opt}) = 0$, and the first non-zero higher order derivative denoted by $n$, i.e. $f^n(x) = d^n f / dx^n$, then:

(i) If $n$ is odd, then $x_{opt}$ is an inflection point (neither min nor max)
(ii) If $n$ is even, then $x_{opt}$ is a local optimum and if $f^n(x_{opt}) > 0$ then $x_{opt}$ is a local minimum or if $f^n(x_{opt}) < 0$ then $x_{opt}$ is a local maximum.

These are the necessary and sufficient conditions for determining the state of the stationary point.

**Example 8.5**

Consider the function given by

$$f(x) = 5x^6 - 36x^5 + \frac{165}{2}x^4 - 60x^3 + 36 \quad (8.42)$$

Find all the stationary points and determine whether they are minimum, maximum, or inflection points.

**8.4.2 Multi-variable Functions**

In the case of a multi-variable function, the procedure is somewhat more complicated. We must determine the eigenvalues of the Hessian matrix which is evaluated at the stationary point (the potential optimal solution). Recall that the Hessian matrix is defined as:

$$H(x) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix} \quad (8.43)$$

If we denote the stationary point of a multi-variable function:

$$x^* = [x_1, x_2, \cdots, x_n]_{opt} \quad (8.44)$$

then the eigenvalues of the Hessian matrix are determined by solving the following equation:

$$\det(H(x^*) - \lambda I) = 0 \quad (8.45)$$
where $\lambda$ are the eigenvalues and $I$ is the identity matrix. The determinant of the $n \times n$ matrix yields an $n^{th}$ order polynomial equation in $\lambda$ which has $n$ real roots or eigenvalues.

Once the eigenvalues have been determined, the following criteria determine whether we have a minimum, maximum, or saddle point (multi-dimensional inflection point). If $\lambda_i$ are the eigenvalues of the Hessian matrix the the following apply:

i) $H(x^*)$ is positive definite if all $\lambda_i > 0$ and the stationary point is a minimum

ii) $H(x^*)$ is positive semi-definite if all $\lambda_i \geq 0$ with at least one $\lambda_i = 0$, and the stationary point is a minimum

iii) $H(x^*)$ is negative definite if all $\lambda_i < 0$ and the stationary point is a maximum

iv) $H(x^*)$ is negative semi-definite if all $\lambda_i \leq 0$ with at least one $\lambda_i = 0$, and the stationary point is a maximum

v) $H(x^*)$ is indefinite if some $\lambda_i < 0$ and $\lambda_i > 0$, and the stationary point is a saddle point

Example 8.6

Given the multi-variable function

$$f(x, y) = \frac{2}{xy} + x^3 + 4y^2 + 2x^2 - xy + 2x - 4y$$

(8.46)

determine the stationary point using Newton-Raphson iteration and determine whether it is a minimum, maximum, or saddle point.

8.5 Unconstrained Optimization Problems

Having developed some of the basic elements of optimization, we will now pursue some fundamental applications. These include but are not limited to: optimization of pumping systems, the design of heat exchangers and other devices, multi-stage compression, etc. We shall also, consider other examples in the class notes.

Unconstrained problems are those whose problem statement only contains an objective function. We are essentially optimizing the system performance in such a manner that the optimization variables are free to take on any value since there are no constraints. In other cases, where constraints exist, we may sometimes eliminate all constraints of the equality type, through back substitution into the objective function. This is not always possible. For these other types of problems we require additional methods to be discussed in the next section.

Example 8.7 - Pumping System

Consider the classic problem of pumping fluid between two points through a circular pipe. We desire to find the optimal pipe diameter which minimizes the total cost of the constructing and operating the system, i.e.
\[ C_T = C_c + C_o \quad (8.47) \]

Assume that the cost of pump and piping is of the form:

\[ C_c = C_0 + C_1 D^n L \quad (8.48) \]

and the operation cost of the system is of the form:

\[ C_o = C_2 \frac{\dot{m} \Delta p}{\rho \eta} \quad (8.49) \]

Further, assume that the pressure drop is calculated from:

\[ \Delta p = \frac{4f L}{D} \frac{1}{2} \rho U^2 \quad (8.50) \]

and that

\[ f = 0.046 \text{Re}^{-1/5} D \quad (8.51) \]

The constants \( C_0, C_1, \) and \( C_2 \) are cost indices for the pump, the piping, and the electricity to run the pump. Modify the analysis to consider a pump cost which is proportional to pipe diameter.

**Example 8.8 - Shell and Tube Heat Exchanger**

A shell and tube heat exchanger requires a total length of tubing to be equal to 100 m based on heat transfer needs. The costs of the design of the heat exchanger are as follows:

1) tube cost $900.00
2) shell cost $1100D^{5/2}L
3) cost of space allocated to $320DL

Here \( D \) is the diameter of the shell and \( L \) is the length of the shell. The pre-selected tube arrangement allows for up to 200 tubes/m² inside the shell cross-section. Find the optimal \( D \) and \( L \) of the heat exchanger for minimum cost. Solve the problem as an unconstrained problem. Use a conventional calculus approach and the intersection of asymptotes method. Discuss the accuracy of the two solution approaches.

**Example 8.9 - Multi-stage Compression System**

Consider a multi-stage compression system containing three compressors and two intercoolers. The ideal work of each compression cycle (see Chapter 4), gives:

\[ W_c = \frac{kRT_i}{k-1} \left[ \left( \frac{p_o}{p_i} \right)^{\frac{(k-1)}{k}} - 1 \right] \quad (8.52) \]

If \( p_1 = 100 \text{ kPa} \) and \( p_4 = 1000 \text{ kPa} \), what are the optimal values of \( p_2 \) and \( p_3 \) such that the total work required is a minimum. Assume \( k = 1.4 \) for air and \( T_1 = 298 \text{ K} \). Verify the solution is indeed a minimum.
8.6 Constrained Optimization Problems: Method of Lagrange Multipliers

The general theory for constrained multivariable optimization may be found in Reklaitis et al. (1983), Edgar and Himmelblau (1988), Stoecker (1989) and Winston (1990). The method of Lagrange multipliers may be easily applied to constrained multivariable applications. The general constrained Non-linear Programming (NLP) problem takes the form:

Minimize (or Maximize)

\[ \phi(x_1, x_2, x_3, \ldots, x_n) = 0 \]  \hspace{1cm} (8.53)

Subject to

\[ g_j(x_1, x_2, x_3, \ldots, x_n) = 0 \quad j = 1, \ldots, m \]
\[ h_j(x_1, x_2, x_3, \ldots, x_n) \geq 0 \quad j = m + 1, \ldots, p \]  \hspace{1cm} (8.54)

where \( g_j \) and \( h_j \) are imposed constraints. It is often more convenient to consider the Langrangian form of the NLP in the following manner. A new objective function is defined as follows:

\[
L(x_1 \ldots x_n, \lambda_1 \ldots \lambda_p, \sigma_1 \ldots \sigma_{p-m}) = \phi(x_i) + \sum_{j=1}^{m} \lambda_j g_j(x_i) + \sum_{k=m+1}^{p} \lambda_k (h_k(x_i) - \sigma_k^2)
\]  \hspace{1cm} (8.55)

where \( \lambda_j \) are Lagrange multipliers and \( \sigma_j \) are slack variables. The use of slack variables enables the Lagrange multiplier method to be applied to problems with inequality constraints.

The problem is now reduced to solving the system of equations defined by

\[
\frac{\partial L}{\partial x_i} = 0 \quad i = 1, \ldots, n
\]
\[
\frac{\partial L}{\partial \lambda_j} = 0 \quad j = 1, \ldots, p
\]
\[
\frac{\partial L}{\partial \sigma_j} = 0 \quad k = 1, \ldots, p - m
\]  \hspace{1cm} (8.56)

The above system may be solved using numerical methods such as a the multivariable Newton-Raphson method. The constrained formulation for NLP’s with inequality constraints can become quite complex. Given an NLP with \( n \) variables and \( p \) constraints with \( p - m \) inequality constraints, optimization of the Lagrangian requires simultaneous solution of a system of \( n + 2p - m \) equations. In most problems, the number of constraints prescribed should be judiciously chosen. For example, it is not always necessary to prescribe that all \( x_i > 0 \). In most problems, an optimal solution with \( x_i > 0 \) may be obtained if a reasonable initial guess is made while
leaving the particular $x_i$ unconstrained. While in other problems, constraints such as $x_i < x_c$ may not be necessary if the optimal solution returns $x_i < x_c$ when $x_i$ are unconstrained.

**Example 8.10**

Solve the following problem using Lagrange multipliers:

$$f = x^2 - 2y - y^2$$  \hspace{1cm} (8.57)

subject to:

$$g = x^2 + y^2 - 1 \leq 0$$  \hspace{1cm} (8.58)

**Example 8.11 - Shell and Tube Heat Exchanger**

Re-solve Example 8.8 using the Lagrange multiplier method.
8.7 References