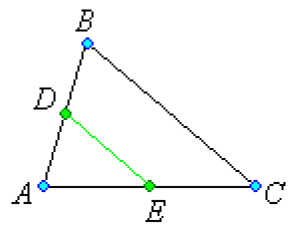
4.4 Additional Examples for Chapter 4

Example 4.4.1

Prove that the line joining the midpoints of two sides of a triangle is parallel to and exactly half as long as the third side of that triangle.

We need to prove that $\overrightarrow{DE} = \frac{1}{2}\overrightarrow{BC}$.



$$\overrightarrow{BC} = \overrightarrow{BO} + \overrightarrow{OC} = \overrightarrow{OC} - \overrightarrow{OB}$$

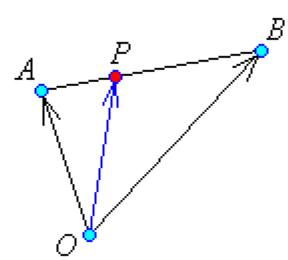
$$\overrightarrow{OD} = \tfrac{1}{2}\left(\overrightarrow{OA} + \overrightarrow{OB}\right) \text{ and } \overrightarrow{OE} = \tfrac{1}{2}\left(\overrightarrow{OA} + \overrightarrow{OC}\right)$$

$$\Rightarrow \quad \overrightarrow{DE} = \overrightarrow{DO} + \overrightarrow{OE} = \overrightarrow{OE} - \overrightarrow{OD} = \tfrac{1}{2}\left(\overrightarrow{OA} + \overrightarrow{OC} - \overrightarrow{OA} - \overrightarrow{OB}\right) = \tfrac{1}{2}\left(\overrightarrow{OC} - \overrightarrow{OB}\right) = \tfrac{1}{2}\overrightarrow{BC}$$

Example 4.4.2

Find the coordinates of the point P that is one-fifth of the way from $A(1, -2, 3)$ to $B(7, 4, -9)$.

$\overrightarrow{OA} = \begin{bmatrix} 1 & -2 & 3 \end{bmatrix}^{\mathrm{T}}$, $\overrightarrow{OB} = \begin{bmatrix} 7 & 4 & -9 \end{bmatrix}^{\mathrm{T}}$



P splits the line segment AB in the ratio $r:s = 1:4$.
The general formula for the location of such a point is

$$\overrightarrow{OP} = \left(\frac{s}{r+s}\right)\overrightarrow{OA} + \left(\frac{r}{r+s}\right)\overrightarrow{OB}$$

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Therefore

$$\overrightarrow{OP} = \left(\frac{4}{5}\right)\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} + \left(\frac{1}{5}\right)\begin{bmatrix} 7 \\ 4 \\ -9 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 11 \\ -4 \\ 3 \end{bmatrix}$$

OR

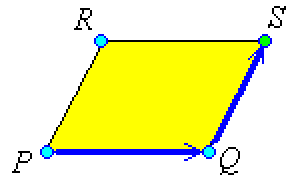
$$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP} = \overrightarrow{OA} + \tfrac{1}{5}\overrightarrow{AB} = \frac{5}{5}\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} + \frac{1}{5}\begin{bmatrix} 6 \\ 6 \\ -12 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 11 \\ -4 \\ 3 \end{bmatrix}$$

Therefore the point P is located at $\left(\frac{11}{5}, -\frac{4}{5}, \frac{3}{5}\right)$.

Example 4.4.3

The points $P(2, 3, 1)$, $Q(4, 7, 2)$, $R(1, 5, 3)$ and S are the four vertices of a parallelogram $PQSR$, with sides PQ and PR meeting at vertex P. Find the coordinates of point S.

Following the path OQS:



$$\overrightarrow{QS} = \overrightarrow{PR} = \overrightarrow{OR} - \overrightarrow{OP} = \begin{bmatrix} -1 & 2 & 2 \end{bmatrix}^T$$

$$\overrightarrow{OS} = \overrightarrow{OQ} + \overrightarrow{QS} = \begin{bmatrix} 4 & 7 & 2 \end{bmatrix}^T + \begin{bmatrix} -1 & 2 & 2 \end{bmatrix}^T = \begin{bmatrix} 3 & 9 & 4 \end{bmatrix}^T$$

Therefore the point S is at (3, 9, 4).

[One could follow the path ORS instead.]

Example 4.4.4

Find the parametric and symmetric equations of the line L that passes through the points $Q(1, -5, 3)$ and $R(4, 7, -1)$. Find the distance r of the point $P(2, -17, 10)$ from the line **and** find the coordinates of the nearest point N on the line to the point P.

The line direction vector is $\vec{\mathbf{d}} = \overrightarrow{QR} = \begin{bmatrix} 3 & 12 & -4 \end{bmatrix}^T$

Either Q or R may serve as the known point on the line.
Choosing Q, the vector equation of the line is

$$\vec{\mathbf{p}} = \overrightarrow{OQ} + t\,\vec{\mathbf{d}} \quad \Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \\ 3 \end{bmatrix} + t \begin{bmatrix} 3 \\ 12 \\ -4 \end{bmatrix}, \quad (t \in \mathbb{R})$$

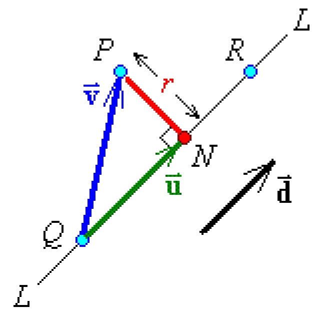
The Cartesian parametric equations are

$$x = 1 + 3t\,, \quad y = -5 + 12t\,, \quad z = 3 - 4t\,, \quad (t \in \mathbb{R})$$

from which the symmetric form follows:

$$\frac{x-1}{3} = \frac{y-(-5)}{12} = \frac{z-3}{-4}$$

Example 4.4.4 (continued)



The vector from Q (a known point on the line) to P is

$$\vec{\mathbf{v}} = \overrightarrow{QP} = \begin{bmatrix} 1 & -12 & 7 \end{bmatrix}^{T}$$

The shadow of this vector on the line is the projection

$$\vec{\mathbf{u}} = \text{proj}_{\vec{\mathbf{d}}}\vec{\mathbf{v}} = \left(\frac{\vec{\mathbf{v}}\bullet\vec{\mathbf{d}}}{\|\vec{\mathbf{d}}\|^2}\right)\vec{\mathbf{d}} = \left(\frac{1}{9+144+16}\right)\left(\begin{bmatrix}1\\-12\\7\end{bmatrix}\bullet\begin{bmatrix}3\\12\\-4\end{bmatrix}\right)\begin{bmatrix}3\\12\\-4\end{bmatrix} = \frac{3-144-28}{169}\begin{bmatrix}3\\12\\-4\end{bmatrix}$$

$$\Rightarrow \quad \vec{\mathbf{u}} = \frac{-169}{169}\begin{bmatrix}3\\12\\-4\end{bmatrix} = -\begin{bmatrix}3\\12\\-4\end{bmatrix}$$

$$\overrightarrow{NP} = \overrightarrow{NQ} + \overrightarrow{QP} = -\vec{\mathbf{u}} + \vec{\mathbf{v}} = \begin{bmatrix}3\\12\\-4\end{bmatrix} + \begin{bmatrix}1\\-12\\7\end{bmatrix} = \begin{bmatrix}4\\0\\3\end{bmatrix}$$

$$\Rightarrow \quad r = \left\|\overrightarrow{NP}\right\| = \sqrt{16+0+9} = \sqrt{25} = 5$$

OR

Triangle PNQ is right-angled at N

$$\Rightarrow \quad r^2 = \|\vec{\mathbf{v}}\|^2 - \|\vec{\mathbf{u}}\|^2 = (1+144+49)-(9+144+16) = 25 \qquad \Rightarrow \quad r = 5$$

The location of N can be found from

$$\overrightarrow{ON} = \overrightarrow{OQ} + \overrightarrow{QN} = \begin{bmatrix}1\\-5\\3\end{bmatrix} + \begin{bmatrix}-3\\-12\\4\end{bmatrix} = \begin{bmatrix}-2\\-17\\7\end{bmatrix}$$

[or one may use $\overrightarrow{ON} = \overrightarrow{OP} + \overrightarrow{PN}$ instead]
Therefore the point N is at (–2, –17, 7).

Example 4.4.5

Show that the lines $L_1 : \frac{x-(-1)}{2} = \frac{y-1}{-1} = \frac{z-2}{3}$ and $L_2 : \frac{x-1}{2} = \frac{y-0}{1} = \frac{z-1}{1}$ are skew **and** find the distance between them.

Line L_1 has line direction vector $\vec{\mathbf{d}}_1 = \begin{bmatrix} 2 & -1 & 3 \end{bmatrix}^T$
and passes through point P_1 (–1, 1, 2).

Line L_2 has line direction vector $\vec{\mathbf{d}}_2 = \begin{bmatrix} 2 & 1 & 1 \end{bmatrix}^T$
and passes through point P_2 (1, 0, 1).

Clearly $\vec{\mathbf{d}}_2$ is not a multiple of $\vec{\mathbf{d}}_1$. Therefore the two lines are not parallel.

OR

The angle between the lines, θ, is also the acute angle between the direction vectors of the lines.

$$\vec{\mathbf{d}}_1 \bullet \vec{\mathbf{d}}_2 = 2\times 2 + (-1)\times 1 + 3\times 1 = 4-1+3 = 6$$

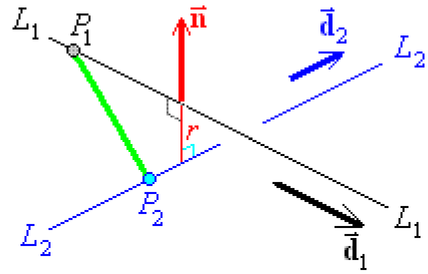
$$\|\vec{\mathbf{d}}_1\| = \sqrt{4+1+9} = \sqrt{14}, \quad \|\vec{\mathbf{d}}_2\| = \sqrt{4+1+1} = \sqrt{6}$$

$$\Rightarrow \quad \cos\theta = \frac{|\vec{\mathbf{d}}_1 \bullet \vec{\mathbf{d}}_2|}{\|\vec{\mathbf{d}}_1\|\|\vec{\mathbf{d}}_2\|} = \frac{6}{\sqrt{14}\sqrt{6}} = \sqrt{\frac{3}{7}} \neq \pm 1 \qquad \Rightarrow \quad \theta \neq 0 \text{ and } \theta \neq \pi$$

The lines are therefore at an angle of $\theta = \cos^{-1}\sqrt{\frac{3}{7}} = \cos^{-1} 0.65465\ldots \approx 49.1°$

Upon finding a non-zero distance between the lines, we will complete the proof that these lines are skew.

Example 4.4.5 (continued)



The vector $\vec{\mathbf{n}} = \vec{\mathbf{d}}_1 \times \vec{\mathbf{d}}_2$ is orthogonal to both lines.

The length of the projection of $\overrightarrow{P_1P_2}$ onto $\vec{\mathbf{n}}$ is the distance r between the lines.

$$\vec{\mathbf{n}} = \begin{vmatrix} \hat{\mathbf{i}} & 2 & 2 \\ \hat{\mathbf{j}} & -1 & 1 \\ \hat{\mathbf{k}} & 3 & 1 \end{vmatrix} = (-1-3)\hat{\mathbf{i}} - (2-6)\hat{\mathbf{j}} + (2+2)\hat{\mathbf{k}} = 4\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$$\overrightarrow{P_1P_2} = \overrightarrow{P_1O} + \overrightarrow{OP_2} = -[\,-1 \quad 1 \quad 2\,]^{\mathrm{T}} + [\,1 \quad 0 \quad 1\,]^{\mathrm{T}} = [\,2 \quad -1 \quad -1\,]^{\mathrm{T}}$$

$$r = \left\| \operatorname{proj}_{\vec{\mathbf{n}}} \overrightarrow{P_1P_2} \right\| = \left| \overrightarrow{P_1P_2} \bullet \hat{\mathbf{n}} \right| = \frac{\left| \overrightarrow{P_1P_2} \bullet \vec{\mathbf{n}} \right|}{\left\| \vec{\mathbf{n}} \right\|} = \frac{\left| \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} \bullet 4 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right|}{4\sqrt{(-1)^2 + 1^2 + 1^2}} = \frac{|-2-1-1|}{\sqrt{1+1+1}} = \frac{4}{\sqrt{3}}$$

Therefore the distance between the two non-parallel lines is $r = \dfrac{4\sqrt{3}}{3} \approx 2.309$

and the two lines are skew.

Example 4.4.6

Find two non-zero vectors that are orthogonal to each other and to $\vec{\mathbf{u}} = [\,3 \quad 2 \quad 0\,]^{\mathrm{T}}$.

It is easy to construct a non-zero vector $\vec{\mathbf{v}}$ whose dot product with $\vec{\mathbf{u}}$ is zero:

$\vec{\mathbf{v}} = [\,0 \quad 0 \quad 1\,]^{\mathrm{T}} \quad \Rightarrow \quad \vec{\mathbf{v}} \bullet \vec{\mathbf{u}} = 0 + 0 + 0 = 0$

For the third vector, just take the cross product of the first two vectors:

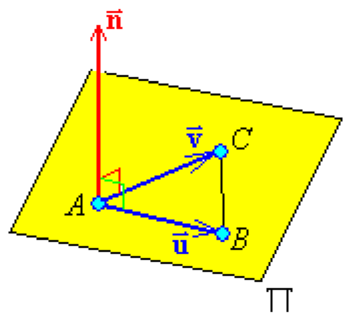
$$\vec{\mathbf{w}} = \vec{\mathbf{u}} \times \vec{\mathbf{v}} = \begin{vmatrix} \hat{\mathbf{i}} & 3 & 0 \\ \hat{\mathbf{j}} & 2 & 0 \\ \hat{\mathbf{k}} & 0 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix}\hat{\mathbf{i}} - \begin{vmatrix} 3 & 0 \\ 0 & 1 \end{vmatrix}\hat{\mathbf{j}} + \begin{vmatrix} 3 & 0 \\ 2 & 0 \end{vmatrix}\hat{\mathbf{k}} = 2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 0\hat{\mathbf{k}}$$

Therefore $\vec{\mathbf{v}} = [\,0 \quad 0 \quad 1\,]^{\mathrm{T}}$ and $\vec{\mathbf{w}} = [\,2 \quad -3 \quad 0\,]^{\mathrm{T}}$.

Example 4.4.7

Find the Cartesian equation of the plane that passes through the points $A(3, 0, 0)$, $B(2, 4, 0)$ and $C(1, 5, 3)$ **and** find the coordinates of the nearest point N on the plane to the origin **and** find the distance of the plane from the origin.

Two vectors in the plane are $\vec{\mathbf{u}} = \overrightarrow{AB} = \begin{bmatrix} -1 & 4 & 0 \end{bmatrix}^T$ and $\vec{\mathbf{v}} = \overrightarrow{AC} = \begin{bmatrix} -2 & 5 & 3 \end{bmatrix}^T$.



A normal to the plane is

$$\vec{\mathbf{u}} \times \vec{\mathbf{v}} = \begin{vmatrix} \hat{\mathbf{i}} & -1 & -2 \\ \hat{\mathbf{j}} & 4 & 5 \\ \hat{\mathbf{k}} & 0 & 3 \end{vmatrix} =$$

$$\begin{vmatrix} 4 & 5 \\ 0 & 3 \end{vmatrix} \hat{\mathbf{i}} - \begin{vmatrix} -1 & -2 \\ 0 & 3 \end{vmatrix} \hat{\mathbf{j}} + \begin{vmatrix} -1 & -2 \\ 4 & 5 \end{vmatrix} \hat{\mathbf{k}} = \begin{bmatrix} 12 \\ 3 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}$$

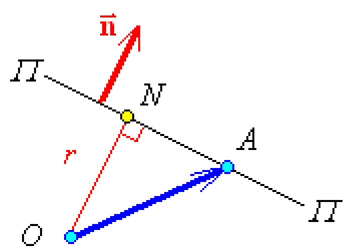
Therefore a normal vector to the plane is $\vec{\mathbf{n}} = \begin{bmatrix} 4 & 1 & 1 \end{bmatrix}^T$

$\vec{\mathbf{a}} = \overrightarrow{OA} = \begin{bmatrix} 3 & 0 & 0 \end{bmatrix}^T \quad \Rightarrow \quad \vec{\mathbf{n}} \bullet \vec{\mathbf{a}} = 3 \times 4 + 0 + 0 = 12$

The Cartesian equation of the plane is

$$4x + y + z = 12$$

The displacement vector for N is the projection of the displacement vector of any point on the plane onto the plane's normal vector.



$$\overrightarrow{ON} = \text{proj}_{\vec{\mathbf{n}}} \overrightarrow{OA} = \left(\overrightarrow{OA} \bullet \hat{\mathbf{n}} \right) \hat{\mathbf{n}} = \left(\frac{\overrightarrow{OA} \bullet \vec{\mathbf{n}}}{\|\vec{\mathbf{n}}\|^2} \right) \vec{\mathbf{n}}$$

$$= \frac{1}{4^2 + 1^2 + 1^2} \left(\begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} \bullet \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix} \right) \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix} = \frac{12}{18} \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix} = \frac{2}{3} \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}$$

Therefore N is the point $\left(\frac{8}{3}, \frac{2}{3}, \frac{2}{3} \right)$ and $r = \left\| \overrightarrow{ON} \right\| = \frac{2}{3}\sqrt{4^2 + 1^2 + 1^2} = \frac{2}{3}\sqrt{18} = 2\sqrt{2}$

Example 4.4.8

Find all vectors $\vec{\mathbf{w}}$ that are orthogonal to both $\vec{\mathbf{u}} = [\,1 \quad 2 \quad 3\,]^T$ and $\vec{\mathbf{v}} = [\,4 \quad 3 \quad 2\,]^T$.

One vector that is orthogonal to both $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ is

$$\vec{\mathbf{u}} \times \vec{\mathbf{v}} = \begin{vmatrix} \hat{\mathbf{i}} & 1 & 4 \\ \hat{\mathbf{j}} & 2 & 3 \\ \hat{\mathbf{k}} & 3 & 2 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} \hat{\mathbf{i}} - \begin{vmatrix} 1 & 4 \\ 3 & 2 \end{vmatrix} \hat{\mathbf{j}} + \begin{vmatrix} 1 & 4 \\ 2 & 3 \end{vmatrix} \hat{\mathbf{k}} = -5\hat{\mathbf{i}} + 10\hat{\mathbf{j}} - 5\hat{\mathbf{k}}$$

Any multiple of this vector is also orthogonal to both $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$.

Therefore the set of vectors $\vec{\mathbf{w}}$ is $\left\{ [\,1 \quad -2 \quad 1\,]^T t\,, \quad (t \in \mathbb{R}) \right\}$.

Check:

$$\vec{\mathbf{w}} \bullet \vec{\mathbf{u}} = [\,1 \quad -2 \quad 1\,]^T t \bullet [\,1 \quad 2 \quad 3\,]^T = (1-4+3)t = 0 \quad \forall t$$

and

$$\vec{\mathbf{w}} \bullet \vec{\mathbf{v}} = [\,1 \quad -2 \quad 1\,]^T t \bullet [\,4 \quad 3 \quad 2\,]^T = (4-6+2)t = 0 \quad \forall t$$

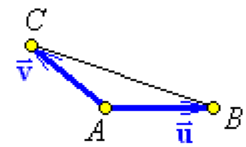
Example 4.4.9

The vertices of a triangle *ABC* are at $A(1, 0, 1)$, $B(-2, -1, 1)$ and $C(3, 2, 2)$.
Find the angle at vertex *A* (correct to the nearest degree).

$$\vec{\mathbf{u}} = \overrightarrow{AB} = [\,-3 \quad -1 \quad 0\,]^T \quad \Rightarrow \quad \|\vec{\mathbf{u}}\| = \sqrt{9+1+0} = \sqrt{10}$$

$$\vec{\mathbf{v}} = \overrightarrow{AC} = [\,2 \quad 2 \quad 1\,]^T \quad \Rightarrow \quad \|\vec{\mathbf{v}}\| = \sqrt{4+4+1} = \sqrt{9} = 3$$

$$\Rightarrow \quad \vec{\mathbf{u}} \bullet \vec{\mathbf{v}} = [\,-3 \quad -1 \quad 0\,]^T \bullet [\,2 \quad 2 \quad 1\,]^T = -6-2+0 = -8$$



Let θ be the angle at vertex *A*, then

$$\cos\theta = \frac{\vec{\mathbf{u}} \bullet \vec{\mathbf{v}}}{\|\vec{\mathbf{u}}\|\,\|\vec{\mathbf{v}}\|} = \frac{-8}{3\sqrt{10}} = -0.84327\ldots$$

$$\Rightarrow \quad \theta = 147.4875\ldots° \approx 147°$$

<u>Example 4.4.10</u> (Textbook, page 179, exercises 4.2, question 18)

Show that every plane containing the points $P(1, 2, -1)$ and $Q(2, 0, 1)$ must also contain the point $R(-1, 6, -5)$.

$\overrightarrow{PQ} = \begin{bmatrix} 1 & -2 & 2 \end{bmatrix}^T$ and $\overrightarrow{QR} = \begin{bmatrix} -3 & 6 & -6 \end{bmatrix}^T = -3\begin{bmatrix} 1 & -2 & 2 \end{bmatrix}^T = -3\,\overrightarrow{PQ}$

Points P and Q are in a plane
$\Rightarrow$ all points on the line through P and Q are in any plane containing P and Q.
But $\overrightarrow{QR} = -3\,\overrightarrow{PQ}$ $\Rightarrow$ point R is on the line through P and Q
Therefore every plane containing the points P and Q must also contain the point R.

<u>Example 4.4.11</u> (Textbook, page 180, exercises 4.2, question 44(a))

Prove the Cauchy-Schwarz inequality $\left|\vec{\mathbf{u}}\bullet\vec{\mathbf{v}}\right| \le \|\vec{\mathbf{u}}\|\|\vec{\mathbf{v}}\|$.

Let θ be the angle between vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$.
$\vec{\mathbf{u}}\bullet\vec{\mathbf{v}} = \|\vec{\mathbf{u}}\|\|\vec{\mathbf{v}}\|\cos\theta$, but $\left|\cos\theta\right| \le 1$
$\Rightarrow$ $\left|\vec{\mathbf{u}}\bullet\vec{\mathbf{v}}\right| \le \|\vec{\mathbf{u}}\|\|\vec{\mathbf{v}}\|$

Example 4.4.12

Find the point of intersection of the lines $\frac{x-3}{2} = \frac{y-3}{1} = \frac{z-0}{-1}$ and $\frac{x-(-1)}{-1} = \frac{y-6}{2} = \frac{z-7}{3}$.

Line 1: $x = 3 + 2s, \quad y = 3 + 1s, \quad z = 0 - 1s$

Line 2: $x = -1 - 1t, \quad y = 6 + 2t, \quad z = 7 + 3t$

At the point of intersection

$x = 3 + 2s = -1 - t \quad \Rightarrow \quad 2s + t = -4$

$y = 3 + s = 6 + 2t \quad \Rightarrow \quad s - 2t = 3$

$z = -s = 7 + 3t \quad \Rightarrow \quad s + 3t = -7$

Solving the over-determined linear system for s and t,

$$\left[\begin{array}{cc|c} 2 & 1 & -4 \\ 1 & -2 & 3 \\ 1 & 3 & -7 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|c} 1 & -2 & 3 \\ 2 & 1 & -4 \\ 1 & 3 & -7 \end{array}\right] \xrightarrow[R_3 - R_1]{R_2 - 2R_1} \left[\begin{array}{cc|c} 1 & -2 & 3 \\ 0 & 5 & -10 \\ 0 & 5 & -10 \end{array}\right]$$

$$\xrightarrow[R_3 - R_2]{} \left[\begin{array}{cc|c} 1 & -2 & 3 \\ 0 & 5 & -10 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow{R_2 \div 5} \left[\begin{array}{cc|c} 1 & -2 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 + 2R_2} \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{array}\right] \quad \Rightarrow \quad s = -1 \text{ and } t = -2$$

Unique solution $\Rightarrow$ a single point of intersection does exist.

$x = 3 + 2(-1) = 1$

$y = 3 + (-1) = 2$

$z = -(-1) = 1$

Therefore the two lines intersect at the point (1, 2, 1).