ENGI 2422 Engineering Mathematics 2
Possibilities for your Formula Sheets

You may select items from this document for placement on your formula sheets.
However, designing your own formula sheet can be a valuable revision exercise in itself.

1. Fundamentals

Equation of a plane, through point P , (where **a** = position vector of P), with non-zero normal vector $\vec{\mathbf{n}} = \langle A, B, C \rangle$:

$$\vec{\mathbf{r}} \bullet \vec{\mathbf{n}} = \vec{\mathbf{a}} \bullet \vec{\mathbf{n}} \quad \text{or} \quad Ax + By + Cz + D = 0$$

Equation of a line, through point P (x_o, y_o, z_o), (where **a** = position vector of P), parallel to non-zero vector $\vec{\mathbf{v}} = \langle v_1, v_2, v_3 \rangle$:

$$\vec{\mathbf{r}} = \vec{\mathbf{a}} + t\,\vec{\mathbf{v}} \quad \text{or} \quad \frac{x - x_o}{v_1} = \frac{y - y_o}{v_2} = \frac{z - z_o}{v_3}$$

If $v_1 = 0$, then separate out the equation $x = x_o$.
If $v_2 = 0$, then separate out the equation $y = y_o$.
If $v_3 = 0$, then separate out the equation $z = z_o$.

The unit tangent, unit principal normal and binormal vectors at any point on a curve given by $\mathbf{r} = \mathbf{r}(t)$ are

$$\hat{\mathbf{T}} = \frac{d\vec{\mathbf{r}}}{dt} \div \left| \frac{d\vec{\mathbf{r}}}{dt} \right|, \qquad \hat{\mathbf{N}} = \frac{d\hat{\mathbf{T}}}{dt} \div \left| \frac{d\hat{\mathbf{T}}}{dt} \right| \qquad \text{and} \qquad \hat{\mathbf{B}} = \hat{\mathbf{T}} \times \hat{\mathbf{N}}$$

The arc length s along the curve can be found from

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} = \left| \frac{d\vec{\mathbf{r}}}{dt} \right|$$

The curvature κ is

$$\kappa = \left| \vec{\mathbf{N}} \right| = \left| \frac{d\hat{\mathbf{T}}}{ds} \right| = \left| \frac{d\hat{\mathbf{T}}}{dt} \right| \div \left| \frac{d\vec{\mathbf{r}}}{dt} \right| = \frac{\left| \dot{\vec{\mathbf{r}}} \times \ddot{\vec{\mathbf{r}}} \right|}{\left| \dot{\vec{\mathbf{r}}} \right|^3}$$

Conic Sections

$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$: ellipse, major axis = $2a$, minor axis = $2b$, $b = a\sqrt{1-e^2}$, $0 < e < 1$,
foci at $(\pm ae, 0)$. ($b = a$ is a circle)

$\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$: hyperbola, vertices at $(\pm 2a, 0)$, asymptotes $y = \pm bx/a$, $e > 1$,
foci at $(\pm ae, 0)$.

$y^2 = 4ax$: parabola, vertex at $(0, 0)$, focus at $(a, 0)$, $e = 1$.

$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 0$ is a point at $(0, 0)$; $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 0$ is the line pair $y = \pm bx/a$.

Quadric Surfaces

$\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2} = 1$: ellipsoid (special cases are spheroid and sphere)

$\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-\dfrac{z^2}{c^2} = 1$: hyperboloid of one sheet, aligned along z axis

$\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}-\dfrac{z^2}{c^2} = 1$: hyperboloid of two sheets, aligned along x axis

$\dfrac{z}{c}=\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}$: elliptic paraboloid

$\dfrac{z}{c}=\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}$: hyperbolic paraboloid

$\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=0$: A single point at the origin. $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=-1$: Nothing

$\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-\dfrac{z^2}{c^2}=0$: Elliptic cone, aligned along the z axis;
[asymptote to both types of hyperboloid].

$\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$: Elliptic cylinder, aligned along the z axis.

$\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$: Hyperbolic cylinder, aligned along the z axis.

$\dfrac{y}{b}=\dfrac{x^2}{a^2}$: Parabolic cylinder, vertex line on the z axis.

$\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=0$: Line (the z axis) $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=-1$: Nothing

$\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=0$: Plane pair (intersecting along the z axis) $\dfrac{x^2}{a^2}=1$: Parallel Plane Pair

$\dfrac{x^2}{a^2}=0$: Single Plane (the y-z coordinate plane) $\dfrac{x^2}{a^2}=-1$: Nothing

Surfaces of Revolution

$y = f(x)$ rotated around $y = c$.

Equation of surface generated is $(y-c)^2 + z^2 = \left(f(x) - c\right)^2$

Area of curved surface is $A = 2\pi\int_a^b \left| f(x) - c \right| \sqrt{1 + \left(f'(x)\right)^2}\, dx$.

Trigonometric identities

$e^{j\theta} = \cos\theta + j\sin\theta$

$\cos x = \dfrac{e^{jx} + e^{-jx}}{2} = \cosh jx$

$\sin x = \dfrac{e^{jx} - e^{-jx}}{2j} = -j \sinh jx$

$\tan x = \sin x / \cos x$
$\sec x = 1 / \cos x$
$\csc x = 1 / \sin x$
$\cot x = 1 / \tan x$
$\cos(-x) = +\cos x$
$\sin(-x) = -\sin x$
$\tan(-x) = -\tan x$

$\cos^2 x + \sin^2 x = 1$
$\sec^2 x = 1 + \tan^2 x$
$\csc^2 x = 1 + \cot^2 x$

$\dfrac{d}{dx}(\cos x) = -\sin x$

$\dfrac{d}{dx}(\sin x) = \cos x$

$\dfrac{d}{dx}(\tan x) = \sec^2 x$

$\dfrac{d}{dx}(\sec x) = \sec x \tan x$

$\dfrac{d}{dx}(\csc x) = -\csc x \cot x$

$\dfrac{d}{dx}(\cot x) = -\csc^2 x$

$\cos(A+B) = \cos A \cos B - \sin A \sin B$
$\cos 2x = \cos^2 x - \sin^2 x$
$\quad = 2\cos^2 x - 1 = 1 - 2\sin^2 x$
$\sin(A+B) = \sin A \cos B + \cos A \sin B$
$\sin 2x = 2 \sin x \cos x$

Hyperbolic f^n identities

$e^{x} = \cosh x + \sinh x$

$\cosh x = \dfrac{e^{x} + e^{-x}}{2} = \cos jx$

$\sinh x = \dfrac{e^{x} - e^{-x}}{2} = -j \sin jx$

$\tanh x = \sinh x / \cosh x$
$\text{sech}\, x = 1 / \cosh x$
$\text{csch}\, x = 1 / \sinh x$
$\coth x = 1 / \tanh x$
$\cosh(-x) = +\cosh x$
$\sinh(-x) = -\sinh x$
$\tanh(-x) = -\tanh x$

$\cosh^2 x - \sinh^2 x = 1$
$\text{sech}^2 x = 1 - \tanh^2 x$
$\text{csch}^2 x = \coth^2 x - 1$

$\dfrac{d}{dx}(\cosh x) = +\sinh x$

$\dfrac{d}{dx}(\sinh x) = \cosh x$

$\dfrac{d}{dx}(\tanh x) = \text{sech}^2 x$

$\dfrac{d}{dx}(\text{sech}\, x) = -\text{sech}\, x \tanh x$

$\dfrac{d}{dx}(\text{csch}\, x) = -\text{csch}\, x \coth x$

$\dfrac{d}{dx}(\coth x) = -\text{csch}^2 x$

$\cosh(A+B) = \cosh A \cosh B + \sinh A \sinh B$
$\cosh 2x = \cosh^2 x + \sinh^2 x$
$\quad = 2\cosh^2 x - 1 = 1 + 2\sinh^2 x$
$\sinh(A+B) = \sinh A \cosh B + \cosh A \sinh B$
$\sinh 2x = 2 \sinh x \cosh x$

Trigonometric identities (cont'd)

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$$

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

$\sin A \cos B = (\sin(A+B) + \sin(A-B)) / 2$
$\cos A \sin B = (\sin(A+B) - \sin(A-B)) / 2$
$\cos A \cos B = (\cos(A+B) + \cos(A-B)) / 2$
$\sin A \sin B = (\cos(A-B) - \cos(A+B)) / 2$

$$\sin P + \sin Q = 2 \sin \frac{P+Q}{2} \cos \frac{P-Q}{2}$$

$$\sin P - \sin Q = 2 \cos \frac{P+Q}{2} \sin \frac{P-Q}{2}$$

$$\cos P + \cos Q = 2 \cos \frac{P+Q}{2} \cos \frac{P-Q}{2}$$

$$\cos P - \cos Q = -2 \sin \frac{P+Q}{2} \sin \frac{P-Q}{2}$$

Let $t = \tan(x/2)$, then

$$\sin x = \frac{2t}{1+t^2}, \qquad \cos x = \frac{1-t^2}{1+t^2},$$

$$\tan x = \frac{2t}{1-t^2}$$

Some Integrals

$$\int u^n \, du = \begin{cases} \dfrac{u^{n+1}}{n+1} + C & (n \neq -1) \\ \ln|u| + C & (n = -1) \end{cases}$$

$$\int a^u \, du = \frac{a^u}{\ln a} + C \quad (a \neq 1, a > 0)$$

$$\int e^u \, du = e^u + C$$

$$\int \frac{f'(x)}{f(x)} \, dx = \ln|f(x)| + C$$

$$\int \tan u \, du = \ln|\sec u| + C$$

$$\int \cot u \, du = \ln|\sin u| + C$$

$$\int \sec u \, du = \ln|\sec u + \tan u| + C$$

$$\int \csc u \, du = \ln|\csc u - \cot u| + C$$

$$\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C$$

$$\int \frac{du}{\sqrt{a^2 + u^2}} = \sinh^{-1}\left(\frac{u}{a}\right) + C = \ln\left|u + \sqrt{a^2 + u^2}\right| + C_2$$

$$\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$$

$$\int \frac{du}{a^2 - u^2} = \frac{1}{a}\tanh^{-1}\left(\frac{u}{a}\right) + C = \frac{1}{2a}\ln\left|\frac{u+a}{u-a}\right| + C_2$$

$$\int \sqrt{a^2 + u^2} \, du = \frac{u}{2}\sqrt{a^2 + u^2} + \frac{a^2}{2}\sinh^{-1}\left(\frac{u}{a}\right) + C = \frac{u}{2}\sqrt{a^2 + u^2} + \frac{a^2}{2}\ln\left|u + \sqrt{a^2 + u^2}\right| + C_2$$

$$\int \sqrt{a^2 - u^2} \, du = \frac{u}{2}\sqrt{a^2 - u^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{u}{a}\right) + C$$

Integration by parts: $\int u \cdot \frac{dv}{dx}\,dx = \left[\,u \cdot v\,\right] - \int \frac{du}{dx} \cdot v\,dx$

[or tabular format**]**

Some forms that can be obtained from integration by parts:

$$\int \ln u\;du = u\left(\ln u - 1\right) + C$$

$$\int e^{au}\sin bu\;du = \frac{e^{au}}{a^2+b^2}\left(a\sin bu - b\cos bu\right) + C$$

$$\int e^{au}\cos bu\;du = \frac{e^{au}}{a^2+b^2}\left(a\cos bu + b\sin bu\right) + C$$

$$\begin{aligned}\int \sin^m u\,\cos^n u\;du &= \frac{1}{m+n}\left(-\sin^{m-1}u\,\cos^{n+1}u + (m-1)\int \sin^{m-2}u\,\cos^n u\;du\right) \\ &= \frac{1}{m+n}\left(+\sin^{m+1}u\,\cos^{n-1}u + (n-1)\int \sin^m u\,\cos^{n-2}u\;du\right)\end{aligned} \qquad \left(m, n \ge 1\right)$$

Any other anti-derivatives that are required in a question but that cannot be obtained from the identities above will be supplied either directly or by means of a hint in the question.

Leibnitz diff'n of an integral:

$$\frac{d}{dx}\int_{y=f(x)}^{y=g(x)} H(x,y)\,dy =$$

$$H(x,g(x))\cdot\frac{dg}{dx} - H(x,f(x))\cdot\frac{df}{dx} + \int_{y=f(x)}^{y=g(x)}\left(\frac{\partial}{\partial x}H(x,y)\right)dy$$

2. Partial Differentiation

Chain rule: If $y = f(x_1, x_2, ..., x_n)$ and $x_i = g_i(t_1, t_2, ..., t_m)$ then

$$\frac{\partial y}{\partial t_j} = \sum_{i=1}^{n} \frac{\partial y}{\partial x_i}\frac{\partial x_i}{\partial t_j} = \frac{\partial y}{\partial x_1}\frac{\partial x_1}{\partial t_j} + \frac{\partial y}{\partial x_2}\frac{\partial x_2}{\partial t_j} + \cdots + \frac{\partial y}{\partial x_n}\frac{\partial x_n}{\partial t_j}$$

and

$$dy = \sum_{i=1}^{n} \frac{\partial y}{\partial x_i} dx_i = \frac{\partial y}{\partial x_1} dx_1 + \frac{\partial y}{\partial x_2} dx_2 + \cdots + \frac{\partial y}{\partial x_n} dx_n$$

Gradient:

In $\mathbb{R}^3$, $\vec{\nabla} = \hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}$ and $\vec{\nabla} f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$

Rate of change of f in the direction of $\mathbf{a}$ at point P is the directional derivative

$D_{\vec{\mathbf{a}}} f\big|_P = \vec{\nabla} f \bullet \hat{\mathbf{a}}\big|_P$

Jacobian (implicit method):

Conversion from $\{ x_1, x_2, ..., x_n \}$ to $\{ u_1, u_2, ..., u_n \}$ defined implicitly by n equations $f_i(x_1, x_2, ..., x_n, u_1, u_2, ..., u_n) = 0$.

Find all n differentials df_i, then construct the matrix equation

$$A \begin{bmatrix} dx_1 \\ dx_2 \\ \vdots \\ dx_n \end{bmatrix} = B \begin{bmatrix} du_1 \\ du_2 \\ \vdots \\ du_n \end{bmatrix}. \qquad \text{The Jacobian is } \frac{\det B}{\det A}.$$

Jacobian (explicit method):

$$\text{Jacobian} = \frac{\partial(x_1, x_2, \cdots, x_n)}{\partial(u_1, u_2, \cdots, u_n)} = ABS\left(\det \begin{bmatrix} \frac{\partial x_1}{\partial u_1} & \frac{\partial x_1}{\partial u_2} & \cdots & \frac{\partial x_1}{\partial u_n} \\ \frac{\partial x_2}{\partial u_1} & \frac{\partial x_2}{\partial u_2} & \cdots & \frac{\partial x_2}{\partial u_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial u_1} & \frac{\partial x_n}{\partial u_2} & \cdots & \frac{\partial x_n}{\partial u_n} \end{bmatrix} \right)$$

<u>Max-Min</u>:
Check all points:
- on the domain boundary;
- where f is undefined;
- where ∇f is undefined;
- where $\nabla f = \mathbf{0}$.

Second derivative test (at points where $\nabla f = \mathbf{0}$):

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$$

$D > 0$ and $f_{xx} > 0 \Rightarrow$ local minimum
$D > 0$ and $f_{xx} < 0 \Rightarrow$ local maximum
$D < 0 \Rightarrow$ saddle point
$D = 0$: test fails.

<u>Lagrange Multipliers</u>:

Identify function $f(x_1, x_2, \ldots, x_n)$ to be maximized or minimized.
Identify constraint(s) $g(x_1, x_2, \ldots, x_n) = k$.
Solve the system of equations

$$\nabla f = \lambda \nabla g \quad \text{and} \quad g = k .$$

Solution with smallest (largest) value of f is the minimum (maximum).

3. First Order ODEs

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

Separable if $M(x, y) = f(x)\cdot g(y)$ **and** $N(x, y) = u(x)\cdot v(y)$

Linear:

$\dfrac{dy}{dx} + P(x)\cdot y = R(x)$; solution $y = e^{-h}\left(\int e^h R\,dx + C\right)$, where $h = \int P\,dx$

Bernoulli: [not in this semester]

$\dfrac{dy}{dx} + P(x)\cdot y = R(x)\cdot y^u$;

reduce to linear $\dfrac{dw}{dx} + (1-u)\cdot P(x)\cdot w = R(x)$ using $w = \dfrac{y^{1-u}}{1-u}$

Exact if $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$; solution $u(x, y) = c$ where $u = \int M\,dx = \int N\,dy$

Integrating Factor:

Use $I(x)$ to try to make $P(x, y)\,dx + Q(x, y)\,dy = 0$ exact:

$\rightarrow \quad \ln I(x) = \int \dfrac{1}{Q}\left(\dfrac{\partial P}{\partial y} - \dfrac{\partial Q}{\partial x}\right)dx$ (invalid if the integrand is dependent on y).

or

Use $I(y)$ to try to make $P(x, y)\,dx + Q(x, y)\,dy = 0$ exact:

$\rightarrow \quad \ln I(y) = \int \dfrac{1}{P}\left(\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}\right)dy$ (invalid if the integrand is dependent on x).

<u>Reduction of order</u> (missing y term):

To solve $\dfrac{d^2 y}{dx^2} + P(x)\dfrac{dy}{dx} = R(x)$,

Replace $\dfrac{dy}{dx}$ by p and replace $\dfrac{d^2 y}{dx^2}$ by $\dfrac{dp}{dx}$

Reduction of order (missing x term):

To solve $\dfrac{d^2 y}{dx^2} + P\dfrac{dy}{dx} + Q\,y = R$,

Replace $\dfrac{dy}{dx}$ by p and replace $\dfrac{d^2 y}{dx^2}$ by $p\dfrac{dp}{dy}$

4. Second Order Linear ODEs

$$\frac{d^2 y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)\,y = R(x)$$

[P and Q both constant]:
Auxiliary equation:
Solve $\lambda^2 + P\lambda + Q = 0 \qquad \lambda = \lambda_1, \lambda_2$

Complementary function:

Real distinct roots (over-damped):
$y_c = A e^{\lambda_1 x} + B e^{\lambda_2 x}$
Real repeated roots (critically damped):
$y_c = (A + Bx)e^{\lambda x}$
Complex conjugate pair of roots ($\lambda = a \pm bj$) (under-damped):
$y_c = e^{ax}\left(A e^{jbx} + B e^{-jbx}\right)$
$= e^{ax}\left(C\cos bx + D\sin bx\right)$

Particular solution by undetermined coefficients:
If $R(x) = e^{kx}$, then try $y_P = c\,e^{kx}$
If $R(x)$ = (a polynomial of degree n),
then try y_P = (a polynomial of degree n), with all (n + 1) coefficients to be determined.
If $R(x)$ = (a multiple of $\cos kx$ and/or $\sin kx$),
then try $y_P = c\cos kx + d\sin kx$
But: if part (or all) of y_P is included in the C.F., then multiply y_P by x.

Particular solution by variation of parameters:
Let $y_c = A\,y_1 + B\,y_2$ then find

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}, \quad W_1 = \begin{vmatrix} 0 & y_2 \\ R & y_2' \end{vmatrix} = -y_2 R, \quad W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & R \end{vmatrix} = +y_1 R,$$

$$u' = \frac{W_1}{W} \rightarrow u, \quad v' = \frac{W_2}{W} \rightarrow v, \quad \text{then}$$

$$y_P = u \cdot y_1 + v \cdot y_2$$

General solution:
$y = y_c + y_P$
Initial (or boundary) conditions → complete solution.

or use Laplace transforms.

5. Some Inverse Laplace Transforms

$F(s)$	$f(t)$
$\int_0^\infty e^{-st} f(t)\, dt$	$f(t)$
$\dfrac{1}{s^n} \quad (n \in \mathbb{N})$	$\dfrac{t^{n-1}}{(n-1)!}$
$\dfrac{1}{\sqrt{s}}$	$\dfrac{1}{\sqrt{\pi t}}$
$\dfrac{1}{s-a}$	e^{at}
$\dfrac{1}{(s-a)^n} \quad (n \in \mathbb{N})$	$\dfrac{t^{n-1} e^{at}}{(n-1)!}$
e^{-as}	$\delta(t-a)$
$\dfrac{e^{-as}}{s}$	$H(t-a)$
$\dfrac{1}{s^2+\omega^2}$	$\dfrac{\sin \omega t}{\omega}$
$\dfrac{1}{(s-a)^2+\omega^2}$	$\dfrac{e^{at} \sin \omega t}{\omega}$
$\dfrac{1}{(s-a)^2-b^2}$	$\dfrac{e^{at} \sinh bt}{b}$
$\dfrac{(s-a)}{(s-a)^2+\omega^2}$	$e^{at} \cos \omega t$
$\dfrac{(s-a)}{(s-a)^2-b^2}$	$e^{at} \cosh bt$

$F(s)$	$f(t)$
$\dfrac{1}{s\left(s^2+\omega^2\right)}$	$\dfrac{1-\cos \omega t}{\omega^2}$
$\dfrac{1}{s^2\left(s^2+\omega^2\right)}$	$\dfrac{\omega t - \sin \omega t}{\omega^3}$
$\dfrac{1}{\left(s^2+\omega^2\right)^2}$	$\dfrac{\sin \omega t - \omega t \cos \omega t}{2\omega^3}$
$\dfrac{s}{\left(s^2+\omega^2\right)^2}$	$\dfrac{t \sin \omega t}{2\omega}$
$\dfrac{s^2-\omega^2}{\left(s^2+\omega^2\right)^2}$	$t \cos \omega t$
$\dfrac{1}{s} \tanh\left(\dfrac{as}{2}\right)$	Square wave, period $2a$, amplitude 1
$\dfrac{1}{s^2} \tanh\left(\dfrac{as}{2}\right)$	Triangular wave, period $2a$, amplitude a
$\dfrac{b}{as^2} - \dfrac{b}{s\left(e^{as}-1\right)}$	Sawtooth wave, period a, amplitude b
$\{ s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - s^{n-3} f''(0) - \ldots - s f^{(n-2)}(0) - f^{(n-1)}(0) \}$	$\dfrac{d^n f}{dt^n}$
$\dfrac{1}{s} F(s)$	$\int_0^t f(\tau)\, d\tau$
$\dfrac{dF}{ds}$	$-t\, f(t)$

First shift theorem: $\left(\text{with } F(s) = \mathcal{L}\{f(t)\}\right)$

The inverse Laplace transform of $F(s-b)$ is $e^{bt} f(t)$.

Second shift theorem:

The inverse Laplace transform of $e^{-as} F(s)$ is $f(t-a)\, H(t-a)$.

Scaling property (an extension of the first shift theorem)**:**

The inverse Laplace transform of $F(as-b)$ is $\dfrac{1}{a} e^{\frac{bt}{a}} f\left(\dfrac{t}{a}\right)$.

Periodic function:

If $f(t)$ is a periodic function of fundamental period T , then

the Laplace transform of $f(t)$ is $\dfrac{1}{1-e^{-sT}} \displaystyle\int_0^T e^{-st} f(t)\, dt$.

Sifting property of the Dirac delta function:

$$\int_c^d f(t)\, \delta(t-a)\, dt = \begin{cases} f(a) & c \le a \le d \\ 0 & a < c \text{ or } a > d \end{cases}$$

Convolution:

If the Laplace transforms of functions $f(t)$ and $g(t)$ are $F(s)$ and $G(s)$ respectively, then the inverse Laplace transform of $H(s) = F(s) \times G(s)$ is the convolution

$$\begin{aligned} h(t) &= (f * g)(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau = \int_0^t f(t-\tau)\, g(\tau)\, d\tau \\ &= (g * f)(t) \end{aligned}$$

Also:

$$\int_0^s F(\sigma)\, d\sigma = \int_0^\infty \left(1-e^{-st}\right)\frac{f(t)}{t}\, dt \Rightarrow \int_0^\infty F(s)\, ds = \int_0^\infty \frac{f(t)}{t}\, dt$$

6. Multiple Integration

If the surface density is $\sigma = f(x, y)$, then the <u>mass</u> is

$$m = \int_{x=a}^{b} \left(\int_{y=g(x)}^{h(x)} f(x,y)\, dy \right) dx = \int_{y=c}^{d} \left(\int_{x=p(y)}^{q(y)} f(x,y)\, dx \right) dy ,$$

where the inner integral must be evaluated first.

Polar coordinates: $(x, y) = (r \cos \theta, r \sin \theta)$ and $dA = dx\, dy = r\, dr\, d\theta$.

<u>Centre of mass</u> is at $(\bar{x}, \bar{y})$, where $m\bar{x} = M_y$ and $m\bar{y} = M_x$,

$$m = \iint_D \sigma\, dA , \quad M_x = \iint_D y\, \sigma\, dA \quad \text{and} \quad M_y = \iint_D x\, \sigma\, dA .$$

Cylindrical polar coordinates:

$(x, y, z) = (r \cos \phi, r \sin \phi, z)$ and $dV = dx\, dy\, dz = r\, dr\, d\phi\, dz$.

Spherical polar coordinates:

$(x, y, z) = (r \sin \theta \cos \phi, r \sin \theta \sin \phi, r \cos \theta)$

and $dV = dx\, dy\, dz = r^2 \sin\theta\, dr\, d\theta\, d\phi$.

Mass $m = \iiint_V \rho\, dV$.

Additional Formulae for Polar Coordinates (if needed)

$(x, y) = (r \cos \theta,\ r \sin \theta) \Rightarrow r^2 = x^2 + y^2 , \qquad \tan\theta = \dfrac{y}{x}$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

Arc length $L = \displaystyle\int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$

Area swept out by $r = f(\theta)$: $A = \dfrac{1}{2}\displaystyle\int_\alpha^\beta r^2\, d\theta$

$\dot{\hat{\mathbf{r}}} = \dot{\theta}\hat{\boldsymbol{\theta}} , \quad \dot{\hat{\boldsymbol{\theta}}} = -\dot{\theta}\hat{\mathbf{r}} \quad \Rightarrow \quad \vec{\mathbf{v}} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}$

and $\vec{\mathbf{a}} = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{\mathbf{r}} + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\hat{\boldsymbol{\theta}} = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{\mathbf{r}} + \left(\dfrac{1}{r}\dfrac{d}{dt}\left(r^2\dot{\theta}\right)\right)\hat{\boldsymbol{\theta}}$

END OF APPENDIX A