4. Second Order Linear Ordinary Differential Equations

The general second order linear ordinary differential equation is of the form

$$\frac{d^2 y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)\,y = R(x)$$

Of the second (and higher) order ordinary differential equations, only linear equations with constant coefficients will be considered in this chapter:

$$\boxed{\frac{d^2 y}{dx^2} + P\frac{dy}{dx} + Q\,y = R(x)}$$

4.1 Complementary Function

The **homogeneous equation** associated with this ODE is

$$\frac{d^2 y}{dx^2} + P\frac{dy}{dx} + Q\,y = 0$$

The principle of superposition of solutions of the homogeneous equation is valid because it is linear. That is, if $y = u(x)$ and $y = v(x)$ are both solutions of the homogeneous ODE, then so also is $y = c_1\,u(x) + c_2\,v(x)$, where c_1 and c_2 are any constants.
Adding any solution of the homogeneous ODE to a particular solution of the original ODE generates another solution of the original ODE.

Thus the **general solution** (abbreviated as G.S.) of

$$\frac{d^2 y}{dx^2} + P\frac{dy}{dx} + Q\,y = R(x)$$

can be partitioned into two parts:
the **complementary function** (C.F., which is the general solution of the associated homogeneous ODE) and a **particular solution** (P.S.).

If $y = e^{\lambda x}$ is a solution to the homogeneous ODE, then

$$\lambda^2 e^{\lambda x} + P\,\lambda\,e^{\lambda x} + Q\,e^{\lambda x} = 0$$

But $e^{\lambda x} > 0$ for all real λ and x.

from which the **auxiliary equation** (A.E.) follows:

$$\lambda^2 + P\lambda + Q = 0$$

[The choice of $y = e^{\lambda x}$ as a trial solution to the homogeneous ODE is justified later, on page 4-08, when a more general method for finding the complementary function is introduced.]

The solution of the auxiliary equation $\lambda^2 + P\lambda + Q = 0$ is

$$\lambda = \frac{-P \pm \sqrt{P^2 - 4Q}}{2} = \lambda_1, \lambda_2$$

Distinct roots ($\lambda_1 \neq \lambda_2$) $\Rightarrow$ the complementary function is

$$\boxed{y_C(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}}$$

[The case of equal roots will be dealt with later, on page 4.07 .]

<u>Example 4.1.1</u>

Solve the differential equation

$$y'' + 3y' - 4y = 0$$

The auxiliary equation is

$$\lambda^2 + 3\lambda - 4 = 0$$

$$\Rightarrow \quad (\lambda + 4)(\lambda - 1) = 0$$

$$\Rightarrow \quad \lambda = -4, +1.$$

The complementary function (which is also the general solution) is

$$y = y_C = \underline{\underline{c_1 e^{-4x} + c_2 e^x}}$$

Checking the solution:

$$y = c_1 e^{-4x} + c_2 e^x$$

$$\Rightarrow \quad y' = -4 c_1 e^{-4x} + c_2 e^x$$

$$\Rightarrow \quad y'' = +16 c_1 e^{-4x} + c_2 e^x$$

$$\Rightarrow \quad y'' + 3y' - 4y = c_1 e^{-4x}(16 + 3(-4) - 4(1)) + c_2 e^x (1 + 3 - 4) \equiv 0 \quad \checkmark$$

Example 4.1.2

Solve

$$y'' - 2y' + 2y = 0$$

A.E.: $\lambda^2 - 2\lambda + 2 = 0$

$$\Rightarrow \quad \lambda = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm j$$

C.F.:

$$y_C = c_1 e^{(1-j)x} + c_2 e^{(1+j)x} = e^x \left(c_1 e^{-jx} + c_2 e^{+jx}\right)$$

$$= e^x \left(c_1 (\cos x - j \sin x) + c_2 (\cos x + j \sin x)\right)$$

$$\Rightarrow \quad y_C = \underline{\underline{e^x (c_3 \cos x + c_4 \sin x)}}$$

In general, when the roots of the auxiliary equation are a complex conjugate pair of values, $\lambda = a \pm bj$, then the complementary function is

$$\boxed{y_C(x) = e^{ax}\left(c_1 e^{-jbx} + c_2 e^{jbx}\right) = e^{ax}\left(c_3 \cos bx + c_4 \sin bx\right)}$$

(where the arbitrary constants are related by $c_3 = c_1 + c_2$ and $c_4 = j(c_2 - c_1)$)
or

$$y_C(x) = A e^{ax} \cos(bx - \delta)$$

$$\left(\text{where} \quad A = \sqrt{c_3^2 + c_4^2}, \quad \cos\delta = \frac{c_3}{\sqrt{c_3^2 + c_4^2}} \quad \text{and} \quad \sin\delta = \frac{-c_4}{\sqrt{c_3^2 + c_4^2}}\right)$$

or

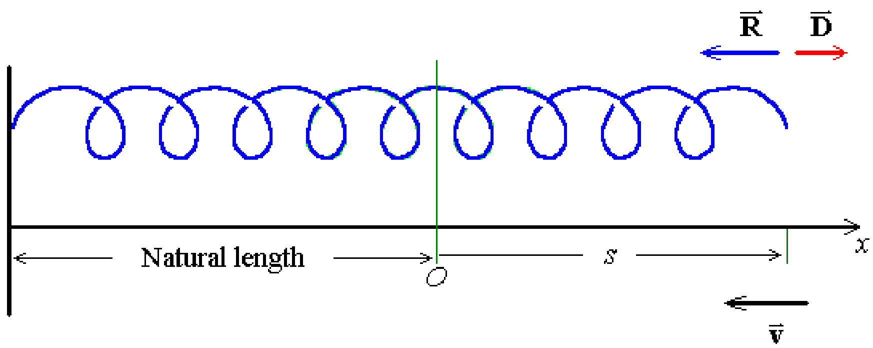
$$y_C(x) = A e^{ax} \sin(bx - \delta)$$

$$\left(\text{where} \quad A = \sqrt{c_3^2 + c_4^2}, \quad \sin\delta = \frac{c_3}{\sqrt{c_3^2 + c_4^2}} \quad \text{and} \quad \cos\delta = \frac{c_4}{\sqrt{c_3^2 + c_4^2}}\right)$$

Note that, for an auxiliary equation of this type, with real coefficients, where the solution is constrained to be real, the arbitrary constants c_3 and c_4 are both real, but c_1 and c_2 often are not. For this reason, the forms involving the trigonometric functions are usually preferred.

Example 4.1.3

A spring, that is not at its natural length, experiences a restoring force **R** that is proportional to the extension s beyond the natural length and is directed towards the equilibrium position. In the absence of friction, this would lead to undamped simple harmonic motion. Let us suppose that there is also a friction force **D** that is proportional to the speed and acts in the opposite direction to the velocity.



Restoring force proportional to displacement $\Rightarrow$

$$R = -c\,s$$

Friction (drag) proportional to speed $\Rightarrow$

$$D = -b\,v$$

Newton's second law of motion:

$$F = \frac{d}{dt}(mv) = m\frac{dv}{dt}$$

Therefore the ODE governing the motion of the spring is

$$m\frac{dv}{dt} = -cs - bv$$

But $v = \dfrac{ds}{dt} \quad \Rightarrow$

$$m\frac{d^2s}{dt^2} = -cs - b\frac{ds}{dt}$$

Therefore

$$s'' + \frac{b}{m}s' + \frac{c}{m}s = 0$$

Example 4.1.3 (continued)

Suppose that $m = 1$ kg, $b = 6$ kg s^{-1} , $c = 25$ kg s^{-2} and that the spring begins at its equilibrium position, but moving at 2 m s^{-1} to the right, so that $s(0) = 0$ and $v(0) = 2$, then the ODE becomes

$$s'' + 6s' + 25s = 0$$

A.E.: $\lambda^2 + 6\lambda + 25 = 0$

$$\lambda = \frac{-6 \pm \sqrt{36-100}}{2} = -3 \pm 4j$$

C.F.: $y_C = s = A\,e^{-3t}\sin(4t-\delta)$

which is **damped harmonic motion**.

Speed:

$$v(t) = s'(t) = A\,e^{-3t}\left(4\cos(4t-\delta) - 3\sin(4t-\delta)\right)$$

Initial conditions:

$$s(0) = 0 \quad\Rightarrow\quad 0 = A\sin(-\delta)$$

$$v(0) = 2 \quad\Rightarrow\quad 2 = A\left(4\cos(-\delta) - 3\sin(-\delta)\right)$$

$$\Rightarrow\quad \delta = 0 \quad \left(\text{or } \delta = n\pi,\ n \in \mathbb{Z}\right)$$

and $A = \frac{2}{4} = \frac{1}{2}$

OR

$$s = e^{-3t}\left(c\cos 4t + d\sin 4t\right)$$

Initial conditions:

$$s(0) = 0 \quad\Rightarrow\quad 0 = c$$

$$s = d\,e^{-3t}\sin 4t$$

$$v = d\,e^{-3t}\left(-3\sin 4t + 4\cos 4t\right)$$

$$v(0) = 2 \quad\Rightarrow$$

$$2 = 4d \quad\Rightarrow\quad d = \tfrac{1}{2}$$

The complete solution, in its simplest form, is

$$s(t) = \underline{\underline{\tfrac{1}{2}e^{-3t}\sin 4t}}$$

Note that if $b = 0$ (no friction at all), then the system is totally undamped and exhibits **simple harmonic motion**:

$$s(t) = A\sin(kt - \delta)$$

where $k = \sqrt{\dfrac{c}{m}}$.

The General Spring Problem

$$\frac{d^2 s}{dt^2} + \frac{b}{m}\frac{ds}{dt} + \frac{c}{m}s = 0$$

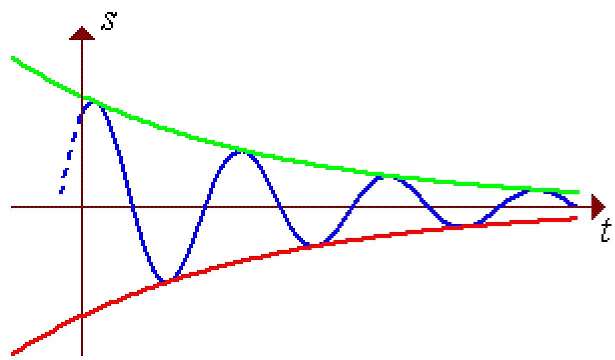
Case 1: $\left(\frac{b}{m}\right)^2 < 4\left(\frac{c}{m}\right)$

λ = complex conjugate pair

$\rightarrow$ damped oscillations.

$s(t) = A\,e^{-at}\sin(kt - \delta)$, where

$a = \frac{-b}{2m}$, $k = \frac{1}{2}\sqrt{\frac{4c}{m} - \left(\frac{b}{m}\right)^2}$



This is the **under-damped** case.

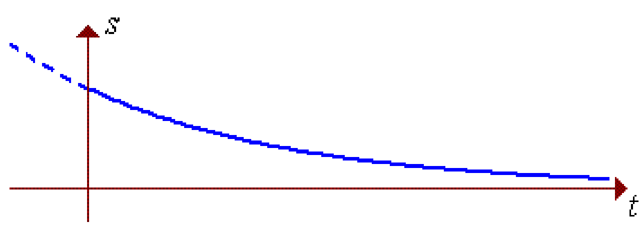
Case 2: $\left(\frac{b}{m}\right)^2 > 4\left(\frac{c}{m}\right)$

λ = distinct real negative pair

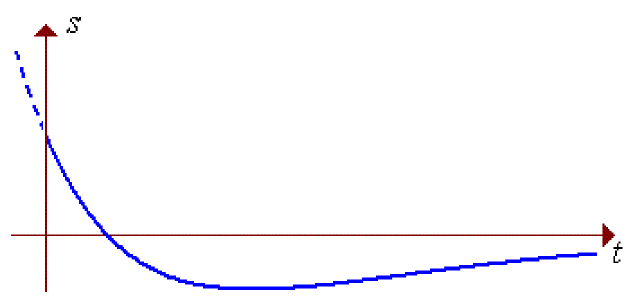
$s(t) = A\,e^{-\lambda_1 t} + B\,e^{-\lambda_2 t}$, where

$\lambda_1, \lambda_2 = \frac{b}{2m} \pm \frac{1}{2}\sqrt{\left(\frac{b}{m}\right)^2 - \frac{4c}{m}}$

This is the **over-damped** case.



or



Case 3: $\left(\frac{b}{m}\right)^2 = 4\left(\frac{c}{m}\right)$

λ = real negative equal roots

This is the **critically damped** case.
[The graphs are similar to those for the over-damped case.]

The solution is $s(t) = (At + B)e^{-\lambda t}$, where $\lambda = \frac{b}{2m}$.

Complementary Function when the Auxiliary Equation has Equal Roots

$\lambda_1 = \lambda_2 (= \lambda) \quad \Rightarrow$ the ODE becomes

$$y'' - 2\lambda y' + \lambda^2 y = 0$$

One solution to this equation is $C_1 e^{\lambda x}$

We require another solution that is independent of this one (so that there will be two distinct arbitrary constants of integration in the complementary function).

Try $f(x) = C_2 x e^{\lambda x}$

[This second form arises naturally from the operator method, on page 4.08.]

Then $f'(x) = C_2 (\lambda x + 1) e^{\lambda x}$

and $f''(x) = C_2 (\lambda^2 x + \lambda + \lambda + 0) e^{\lambda x}$

$\Rightarrow \quad f''(x) - 2\lambda f'(x) + \lambda^2 f(x) =$

$$C_2 (\lambda^2 x + 2\lambda - 2\lambda^2 x - 2\lambda + \lambda^2 x) e^{\lambda x} = 0$$

Therefore $f(x) = C_2 x e^{\lambda x}$ is another solution to the homogeneous ODE.

Therefore the C.F. is

$$\boxed{y_C(x) = (C_1 + C_2 x) e^{\lambda x}}$$

<u>Example 4.1.4</u>

Solve

$$y'' - 6y' + 9y = 0$$

A.E.: $\lambda^2 - 6\lambda + 9 = 0$

$\Rightarrow \quad (\lambda - 3)^2 = 0 \quad \Rightarrow \lambda = 3, 3$

Therefore $\quad y = y_C = \underline{\underline{(Ax + B) e^{3x}}}$

The Operator Method

The homogeneous ordinary differential equation with constant coefficients,

$$\frac{d^2 y}{dx^2} + P\frac{dy}{dx} + Q\,y = 0$$

can also be written, using differential operators, in the form

$$\left(\frac{d}{dx}+k_1\right)\left(\frac{d}{dx}+k_2\right)y = 0$$

Justification:

$$\left(\frac{d}{dx}+k_1\right)\left(\frac{d}{dx}+k_2\right)y = \left(\frac{d}{dx}+k_1\right)\left(\frac{dy}{dx}+k_2 y\right)$$

$$= \left(\frac{d^2 y}{dx^2}+k_2\frac{dy}{dx}\right) + \left(k_1\frac{dy}{dx}+k_1 k_2 y\right)$$

$$= \frac{d^2 y}{dx^2} + (k_1+k_2)\frac{dy}{dx} + k_1 k_2 y = 0$$

$\Rightarrow \quad k_1 + k_2 = P$ and $k_1 k_2 = Q$.

$\Rightarrow -k_1$ and $-k_2$ are the solutions to the auxiliary equation $\lambda^2 + P\lambda + Q = 0$.

The second order ODE can therefore be re-written as a linked pair of first order linear ordinary differential equations [the method of reduction of order]:

$$\left(\frac{d}{dx}+k_1\right)\theta = 0\,, \qquad \textbf{(A)}$$

where

$$\theta = \frac{dy}{dx}+k_2 y\,. \qquad \textbf{(B)}$$

Solution:

(A) is linear

$$\theta' + \underset{\substack{\uparrow \\ P}}{k_1}\,\theta = \underset{\substack{\uparrow \\ R}}{0}$$

$$h = \int P\,dx = k_1\int 1\,dx = k_1 x$$

$$e^h = e^{k_1 x}$$

Operator Method (continued)

$$\int e^h R\,dx = \int 0\,dx = 0$$

$$\Rightarrow \quad \theta(x) = e^{-h}\left(\int e^h R\,dx + C\right) = e^{-k_1 x}(0 + C) = C_1 e^{-k_1 x}$$

OR

(A) is separable

$$\frac{d\theta}{dx} = -k_1\theta \qquad \Rightarrow \quad \int \frac{d\theta}{\theta} = -k_1 \int dx$$

$$\Rightarrow \quad \ln\theta = -k_1 x + C$$

$$\Rightarrow \quad \theta(x) = e^{-k_1 x + C} = e^C e^{-k_1 x} = C_1 e^{-k_1 x}$$

Feed the solution from ODE **(A)** into ODE **(B)**:

$$y' + \underset{\substack{\uparrow \\ P}}{k_2}\, y = \underbrace{C_1 e^{-k_1 x}}_{R}$$

$$h = \int P\,dx = k_2 \int 1\,dx = k_2 x$$

$$e^h = e^{k_2 x}$$

$$\int e^h R\,dx = \int e^{k_2 x}\left(C_1 e^{-k_1 x}\right) dx = C_1 \int e^{(k_2 - k_1)x}\,dx$$

There are two cases to consider:

$$\int e^h R\,dx = C_1 \times \begin{cases} \dfrac{e^{(k_2 - k_1)x}}{k_2 - k_1} & (k_2 \neq k_1) \\[2ex] x & (k_2 = k_1) \end{cases}$$

Operator Method (continued)

$k_2 \neq k_1$:

$$y(x) = e^{-k_2 x}\left(\frac{C_1 e^{(k_2 - k_1)x}}{k_2 - k_1} + C_2\right) = \underline{\underline{A e^{-k_1 x} + B e^{-k_2 x}}}$$

$k_2 = k_1 \ (= k)$:

$$y(x) = e^{-k_2 x}\left(C_1 x + C_2\right) = \underline{\underline{\left(Ax + B\right) e^{-kx}}}$$

<u>Summary for the Complementary Function</u>:

ODE: $y'' + P y' + Q y = 0$

A.E.: $\lambda^2 + P\lambda + Q = 0$

λ real and distinct $\Rightarrow \quad y_C = A e^{\lambda_1 x} + B e^{\lambda_2 x}$

λ real and equal $\Rightarrow \quad y_C = \left(Ax + B\right) e^{\lambda x}$

λ complex conjugate pair

$$\Rightarrow \quad y_C = e^{ax}\left(C\cos bx + D\sin bx\right), \quad \text{where} \quad a = \mathrm{Re}(\lambda),\ b = \left|\mathrm{Im}(\lambda)\right|$$

4.2 Particular Solution (Undetermined Coefficients)

The general solution to

$$\frac{d^2 y}{dx^2} + P\frac{dy}{dx} + Q\,y = R(x)$$

is the sum of the complementary function and any one solution (the particular solution) that we can find to the original inhomogeneous ODE.

If the function $R(x)$ does not contain any part of the complementary function, then assume that the particular solution $y_P(x)$ is of the same form as $R(x)$.

Example 4.2.1

Find the general solution of the ODE $\quad y'' + 2y' - 3y = x^2 + e^{2x}$.

A.E.: $\lambda^2 + 2\lambda - 3 = 0$

$\Rightarrow \quad (\lambda+3)(\lambda-1) = 0 \quad \Rightarrow \quad \lambda = -3, 1$

C.F.: $y_C = A\,e^{-3x} + B\,e^{x}$

P.S.: $R(x)$ contains neither e^{-3x} nor e^{x} .

$R(x)$ is the sum of a quadratic function and e^{2x} .

Therefore try the sum of a quadratic function and a multiple of e^{2x} , where all four coefficients are to be determined.

$$y_P = a\,x^2 + b\,x + c + d\,e^{2x}$$

$$\Rightarrow \quad y_P' = 2a\,x + b + 2d\,e^{2x}$$

$$\Rightarrow \quad y_P'' = 2a + 4d\,e^{2x}$$

$$\Rightarrow \quad y_P'' + 2\,y_P' - 3\,y_P =$$

$$\begin{array}{cccccccccc}
 & & & & & 2a & + & 4d\,e^{2x} \\
+ & & & 4a\,x & + & 2b & + & 4d\,e^{2x} \\
+ & -3a\,x^2 & - & 3b\,x & - & 3c & - & 3d\,e^{2x} \\
\hline
= & 1\,x^2 & + & 0\,x & + & 0 & + & 1\,e^{2x}
\end{array}$$

Example 4.2.1 (continued)

Matching coefficients:

$$x^2: \quad -3a = 1 \quad \Rightarrow \quad a = -\tfrac{1}{3}$$

$$x^1: \quad 4\left(-\tfrac{1}{3}\right) - 3b = 0 \quad \Rightarrow \quad b = -\tfrac{4}{9}$$

$$x^0: \quad 2\left(-\tfrac{1}{3}\right) + 2\left(-\tfrac{4}{9}\right) - 3c = 0 \quad \Rightarrow \quad c = -\frac{2}{3}\left(\frac{3+4}{9}\right) = -\tfrac{14}{27}$$

$$e^{2x}: \quad (4+4-3)d = 1 \quad \Rightarrow \quad d = \tfrac{1}{5}$$

G.S.: $y(x) = y_C(x) + y_P(x)$

Therefore

$$y(x) = \underline{\underline{A e^{-3x} + B e^{x} + \tfrac{1}{5} e^{2x} - \tfrac{1}{27}\left(9x^2 + 12x + 14\right)}}$$

General Method:

The general solution to

$$\frac{d^2 y}{dx^2} + P\frac{dy}{dx} + Q y = R(x)$$

is the sum of the complementary function and any one solution (the particular solution) that we can find to the original inhomogeneous ODE.

If the function $R(x)$ does not contain any part of the complementary function, then assume that the particular solution $y_P(x)$ is of the same form as $R(x)$.

If $R(x) = e^{kx}$,
then try $y_P = c\, e^{kx}$, with c to be determined.

If $R(x) =$ (a polynomial of degree n),
then try $y_P =$ (a polynomial of degree n), with all $(n + 1)$ coefficients to be determined.

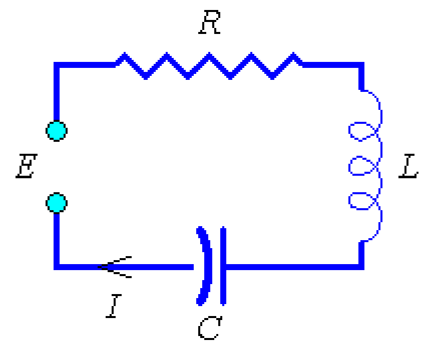
If $R(x) =$ (a multiple of $\cos kx$ and/or $\sin kx$),
then try $y_P = c \cos kx + d \sin kx$, with c and d to be determined.

This method can be extended to cases where
$R(x) =$ (a sum and/or product of the functions above).

But: if part (or all) of y_P is included in the C.F., then multiply y_P by x.

Example 4.2.2

Consider a model of the simple series RLC circuit, where the constants R, L, C are the resistance, inductance and capacitance respectively, $E(t)$ is the applied electromotive force, t is the time and $I(t)$ is the resulting current.



Examine the voltage drops around the circuit:

$R: \quad RI$

$L: \quad L\dfrac{dI}{dt}$

$C: \quad \dfrac{Q}{C}$ and note that $I = \dfrac{dQ}{dt}$.

$$\Rightarrow \quad L\frac{dI}{dt} + RI + \frac{Q}{C} = E$$

$$\Rightarrow \quad \frac{d^2I}{dt^2} + \frac{R}{L}\frac{dI}{dt} + \frac{1}{LC}I = \frac{1}{L}\frac{dE}{dt}$$

A.E.:

$$\lambda^2 + \frac{R}{L}\lambda + \frac{1}{LC} = 0$$

$$\Rightarrow \quad \lambda = \frac{1}{2}\left(-\frac{R}{L} \pm \sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{LC}}\right)$$

Let $D = -\left(R^2 - \dfrac{4L}{C}\right)$, then $\lambda = \dfrac{-R}{2L} \pm j\dfrac{\sqrt{D}}{2L}$

[Note that the numerical value of the capacitance C is usually so minute that it is safe to assume that $D > 0$.]

C.F.:

$$y_C = e^{-Rt/(2L)}\left(A\sin\frac{\sqrt{D}\,t}{2L} + B\cos\frac{\sqrt{D}\,t}{2L}\right)$$

which is the **transient term**.

Example 4.2.2 (continued)

Particular solution

If $E(t) = E_o$ (constant), then

$I(t) = y_C + 0 \Rightarrow \lim_{t \to \infty} I(t) = 0$

Suppose that the e.m.f. is sinusoidal, so that $E(t) = E_o \sin \omega t$, then

$$\frac{1}{L}\frac{dE}{dt} = \frac{E_o \omega \cos \omega t}{L}$$

P.S.: Try
$y_P = a \sin \omega t + b \cos \omega t$

$\Rightarrow \quad y_P' = -b\omega \sin \omega t + a\omega \cos \omega t$

$\Rightarrow \quad y_P'' = -a\omega^2 \sin \omega t - b\omega^2 \cos \omega t$

$$\Rightarrow \quad y_P'' + \frac{R}{L} y_P' + \frac{1}{LC} y_P =$$
$$\left(-a\omega^2 - \frac{b\omega R}{L} + \frac{a}{LC}\right)\sin \omega t + \left(-b\omega^2 + \frac{a\omega R}{L} + \frac{b}{LC}\right)\cos \omega t$$
$$= 0 \sin \omega t + \frac{E_o \omega}{L} \cos \omega t$$

$$\Rightarrow \quad \frac{b\omega R}{L} = \frac{a}{LC} - a\omega^2 \qquad \Rightarrow \quad b = \frac{a}{\omega RC}\left(1 - \omega^2 LC\right)$$

$$\Rightarrow \quad \frac{a\omega R}{L} + \left(\frac{a}{\omega RC}\left(1 - \omega^2 LC\right)\right)\left(-\omega^2 + \frac{1}{LC}\right) = \frac{E_o \omega}{L}$$

$$\Rightarrow \quad \frac{a\omega R\left(\omega RC\, C\right)}{\omega RC\, LC} + \left(\frac{a\left(1 - \omega^2 LC\right)^2}{\omega RC\, LC}\right) = \frac{E_o \omega}{L}$$

$$\Rightarrow \quad a = \frac{E_o \omega}{L} \times \frac{\omega RC\, LC}{\left(\omega RC\right)^2 + \left(1 - \omega^2 LC\right)^2} = \frac{E_o \omega C\left(\omega RC\right)}{\left(\omega RC\right)^2 + \left(1 - \omega^2 LC\right)^2}$$

$$\Rightarrow \quad b = \frac{1 - \omega^2 LC}{\omega RC} \times \frac{E_o \omega C\left(\omega RC\right)}{\left(\omega RC\right)^2 + \left(1 - \omega^2 LC\right)^2} = \frac{E_o \omega C\left(1 - \omega^2 LC\right)}{\left(\omega RC\right)^2 + \left(1 - \omega^2 LC\right)^2}$$

Example 4.2.2 (continued)

Therefore the particular solution is

$$\Rightarrow \quad y_P = \frac{E_o \omega C \left((\omega RC)\sin \omega t + \left(1 - \omega^2 LC\right)\cos \omega t \right)}{(\omega RC)^2 + \left(1 - \omega^2 LC\right)^2}$$

which is a steady-state sinusoidal response to the sinusoidal electromotive force, but with a phase difference of $\delta = \arccos\left(\dfrac{\omega RC}{\sqrt{(\omega RC)^2 + \left(1 - \omega^2 LC\right)^2}} \right)$.

The total current is then

$$I(t) = \underbrace{e^{-\frac{Rt}{2L}} \left(A \sin\left(\frac{\sqrt{D}}{2L} t \right) + B \cos\left(\frac{\sqrt{D}}{2L} t \right) \right)}_{\text{transient}} + \underbrace{\left(\frac{E_o \omega C \left((\omega RC)\sin \omega t + \left(1 - \omega^2 LC\right)\cos \omega t \right)}{(\omega RC)^2 + \left(1 - \omega^2 LC\right)^2} \right)}_{\text{steady} - \text{state}}$$

As a specific example, if $E(t) = 17 \sin 2t$, $R = 120\ \Omega$, $C = 1$ mF and $L = 10$ H, then it can be shown that

$$I(t) = e^{-6t}\left(A \sin 8t + B \cos 8t \right) + \frac{1}{120}\left(\sin 2t + 4 \cos 2t \right)$$

The transient current, $I_C(t) = e^{-6t}\left(A \sin 8t + B \cos 8t \right)$, dies away very quickly.
Its magnitude falls permanently to under 1% of the total current in less than a second.
The values of the two arbitrary constants can be found from the initial conditions, but, given that the complementary function becomes negligible in a very short time, one often does not try to evaluate them.

Example 4.2.3

Find the complete solution of the ODE

$$y'' + 2y' + y = e^{-x} \quad , \quad y(0) = y'(0) = 1$$

A.E.: $\lambda^2 + 2\lambda + 1 = 0 \quad \Rightarrow \quad \lambda = -1, -1.$

C.F.: $y_C = (Ax + B)\, e^{-x}$.

Example 4.2.3 (continued)

P.S.: Both $y = e^{-x}$ and $y = x\,e^{-x}$ are included in the complementary function.

Therefore try $y_P = a\,x^2\,e^{-x}$:

$y''_P + 2\,y'_P + y_P = e^{-x} \Rightarrow$

$((2a - 4ax + a\,x^2) + (4ax - 2a\,x^2) + (a\,x^2)\,e^{-x} = 1\,e^{-x}$

$\Rightarrow ((a - 2a + a)\,x^2 + (-4a + 4a)\,x + (2a)\,e^{-x} = 1\,e^{-x}$

$\Rightarrow a = 1/2$

G.S.:

$$y(x) = \left(\tfrac{1}{2}x^2 + Ax + B\right)e^{-x}$$

Now impose the initial conditions on this general solution:

$y(0) = (0 + 0 + B)\,e^0 = 1 \Rightarrow B = 1$

$y'(x) = (x + A - ½x^2 - Ax - B)\,e^{-x}$

$\Rightarrow y'(0) = (0 + A - 0 - 0 - 1)\,e^0 = 1 \Rightarrow A - 1 = 1 \Rightarrow A = 2$

Therefore the complete solution is

$$y(x) = \underline{\underline{\left(\tfrac{1}{2}x^2 + 2x + 1\right)e^{-x}}}$$

Note that a **complete solution** requires additional information (often in the form of **initial conditions**). Two pieces of information are needed in order to evaluate both arbitrary constants of integration. However, do **not** substitute these conditions into the complementary function; wait until the general solution has been obtained.

4.3 Particular Solution (Variation of Parameters)

The method of variation of parameters is a more general method for finding the particular solution. It is successful even in some cases where the method of undetermined coefficients fails. However, where both methods are available, the method of undetermined coefficients is generally faster to use.

If the complementary function for the ODE

$$\frac{d^2 y}{dx^2} + P\frac{dy}{dx} + Q\,y = R(x)$$

is $y_C(x) = C_1\,y_1(x) + C_2\,y_2(x)$,
then the particular solution is

$$y_P(x) = u(x)\,y_1(x) + v(x)\,y_2(x)\,,$$

where the functions $u(x)$ and $v(x)$ need to be determined.

We need two constraints in order to pin down the functional forms of $u(x)$ and $v(x)$.
One constraint is that $u(x)\,y_1(x) + v(x)\,y_2(x)$ be a particular solution of the ODE.
We have considerable freedom as to what the other constraint will be.

$$y_P = u\,y_1 + v\,y_2$$

$$\Rightarrow \quad y'_P = u'\,y_1 + u\,y'_1 + v'\,y_2 + v\,y'_2$$

Impose our "free" constraint, $u'\,y_1 + v'\,y_2 = 0$, then

$$y'_P = u\,y'_1 + v\,y'_2$$

$$\Rightarrow \quad y''_P = u'\,y'_1 + u\,y''_1 + v'\,y'_2 + v\,y''_2$$

$$\Rightarrow \quad y''_P + P\,y'_P + Q\,y_P =$$

$$u\,(y''_1 + P\,y'_1 + Q\,y_1) + v(y''_2 + P\,y'_2 + Q\,y_2) + u'\,y'_1 + v'\,y'_2 = R(x)$$

But $y_1(x)$ and $y_2(x)$ are both solutions to $y'' + P\,y' + Q\,y = 0$

Constraint (2) then resolves to

$$u'\,y'_1 + v'\,y'_2 = R(x)$$

The pair of constraints leads to the matrix equation

$$\begin{bmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{bmatrix}\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} 0 \\ R \end{bmatrix}$$

Define the Wronskian function $W(x)$ to be

$$W_{y_1,y_2}(x) = \det\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}$$

together with the associated determinants

$$W_1 = \det\begin{bmatrix} 0 & y_2 \\ R & y_2' \end{bmatrix} = -y_2 R \quad \text{and} \quad W_2 = \det\begin{bmatrix} y_1 & 0 \\ y_1' & R \end{bmatrix} = +y_1 R$$

then Cramer's rule yields solutions for u' and v':

$$u' = \frac{W_1}{W} \quad \text{and} \quad v' = \frac{W_2}{W}.$$

Therefore a particular solution is $y_P = u\,y_1 + v\,y_2$, where

$$\boxed{u(x) = -\int \frac{y_2(x)R(x)}{W(x)}\,dx, \quad v(x) = +\int \frac{y_1(x)R(x)}{W(x)}\,dx, \quad W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}$$

Note that we can ignore the arbitrary constants of integration in both integrals, because $A\,y_1$ and $B\,y_2$ are both solutions of the homogeneous ODE and can therefore be absorbed into the complementary function.

<u>Example 4.3.1</u> (identical to Example 4.2.1)

Find the general solution of the ODE $\qquad y'' + 2y' - 3y = x^2 + e^{2x}$.

A.E.: $\lambda^2 + 2\lambda - 3 = 0$

$\Rightarrow \qquad (\lambda + 3)(\lambda - 1) = 0 \quad \Rightarrow \quad \lambda = -3, 1$

$y_1 = e^{-3x}, \quad y_2 = e^{x}, \quad R = x^2 + e^{2x}$

Example 4.3.1 (continued)

Particular Solution by Variation of Parameters:

$$W(x) = \det\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} = \det\begin{bmatrix} e^{-3x} & e^{x} \\ -3e^{-3x} & e^{x} \end{bmatrix} = 4e^{-2x}$$

$$W_1 = \det\begin{bmatrix} 0 & y_2 \\ R & y_2' \end{bmatrix} = -y_2 R = -e^{x}\left(x^2 + e^{2x}\right)$$

$$\Rightarrow \quad u' = \frac{W_1}{W} = \frac{-\left(x^2 e^{x} + e^{3x}\right)}{4e^{-2x}} = -\left(\frac{x^2 e^{3x} + e^{5x}}{4}\right)$$

$$\Rightarrow \quad u = -\frac{1}{4}\int\left(x^2 e^{3x} + e^{5x}\right)dx$$

D		**I**
x^2		e^{3x}
	$+$	
$2x$		$\frac{1}{3}e^{3x}$
	$-$	
2		$\frac{1}{9}e^{3x}$
	$+$	
0		$\frac{1}{27}e^{3x}$

$$\Rightarrow \quad u = -\frac{1}{4}\left(\frac{e^{3x}}{27}\left(9x^2 - 6x + 2\right) + \frac{1}{5}e^{5x}\right)$$

$$W_2 = \det\begin{bmatrix} y_1 & 0 \\ y_1' & R \end{bmatrix} = +y_1 R = e^{-3x}\left(x^2 + e^{2x}\right)$$

$$\Rightarrow \quad v' = \frac{W_2}{W} = \frac{x^2 e^{-3x} + e^{-x}}{4e^{-2x}} = \frac{x^2 e^{-x} + e^{x}}{4}$$

$$\Rightarrow \quad v = \frac{1}{4}\int\left(x^2 e^{-x} + e^{x}\right)dx$$

D		**I**
x^2		e^{-x}
	$+$	
$2x$		$-e^{-x}$
	$-$	
2		$+e^{-x}$
	$+$	
0		$-e^{-x}$

$$\Rightarrow \quad v = \frac{1}{4}\left(e^{-x}\left(-x^2 - 2x - 2\right) + e^{x}\right)$$

Example 4.3.1 (continued)

$$y_P = u \cdot y_1 + v \cdot y_2 =$$

$$\frac{1}{4}\left\{\left(-\frac{e^{3x}}{27}\left(9x^2 - 6x + 2\right) - \frac{1}{5}e^{5x}\right)e^{-3x} + \frac{27}{27}\left(-e^{-x}\left(x^2 + 2x + 2\right) + e^{x}\right)e^{x}\right\}$$

$$= \frac{1}{4}\left\{\frac{1}{27}\left(-9x^2 + 6x - 2 - 27x^2 - 54x - 54\right) + \left(-\frac{1}{5} + 1\right)e^{2x}\right\}$$

$$= \frac{1}{4}\left\{\frac{1}{27}\left(-36x^2 - 48x - 56\right) + \frac{4}{5}e^{2x}\right\}$$

Therefore

$$y_P = \frac{1}{5}e^{2x} - \frac{1}{27}\left(9x^2 + 12x + 14\right)$$

and the general solution is

$$y(x) = \underline{\underline{A\,e^{-3x} + B\,e^{x} + \frac{1}{5}e^{2x} - \frac{1}{27}\left(9x^2 + 12x + 14\right)}}$$

Example 4.3.2

Here is a case that cannot be solved by the method of undetermined coefficients:

$$y'' + y = \tan x$$

$R(x) = \tan x$ is not one of the standard forms.

A.E.: $\lambda^2 + 1 = 0$

$\Rightarrow \quad \lambda = \pm j$

C.F.:

$$y_C = A \underbrace{\sin x}_{y_1} + B \underbrace{\cos x}_{y_2}$$

Let $\quad s = \sin x$ and $c = \cos x$

then $\quad y_1 = s \ , \quad y_2 = c \ , \quad y'_1 = c \ , \quad y'_2 = -s$

$$\Rightarrow \quad W = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} = \begin{vmatrix} s & c \\ c & -s \end{vmatrix} = -s^2 - c^2 = -1$$

$$R = \tan x = \frac{s}{c}$$

$$W_1 = \begin{vmatrix} 0 & y_2 \\ R & y'_2 \end{vmatrix} = -y_2 R = -c\left(\frac{s}{c}\right) = -s$$

$$\Rightarrow \quad u' = \frac{W_1}{W} = \frac{-s}{-1} = +s$$

$$\Rightarrow \quad u = \int s \, dx = -c$$

$$W_2 = \begin{vmatrix} y_1 & 0 \\ y'_1 & R \end{vmatrix} = +y_1 R = s\left(\frac{s}{c}\right) = \frac{s^2}{c}$$

$$\Rightarrow \quad v' = \frac{W_2}{W} = \frac{s^2}{-c} = \frac{-\left(1 - c^2\right)}{c} = \cos x - \sec x$$

$$\Rightarrow \quad v = \int \left(\cos x - \sec x\right) dx = \sin x - \ln\left|\sec x + \tan x\right|$$

Example 4.3.2 (continued)

$$\therefore \quad y_P = u \cdot y_1 + v \cdot y_2 = -cs + \left(s - \ln\left|\sec x + \tan x\right|\right) \cdot c$$
$$= -c \ln\left|\sec x + \tan x\right|$$

[which is clearly of a different form from $R(x)$.
It is not a simple linear combination of any trigonometric functions.]

$y = y_C + y_P$

General solution:

$$y(x) = \underline{\underline{A \sin x + \cos x\left(B - \ln\left|\sec x + \tan x\right|\right)}}$$

Example 4.3.3

Use the variation of parameters method to find the particular solution, then find the general solution of the ODE

$$y'' - 2y' + y = e^x$$

A.E.: $\lambda^2 - 2\lambda + 1 = 0$

$\Rightarrow \quad (\lambda - 1)^2 = 0 \quad \Rightarrow \lambda = 1, 1$

C.F.: $y_C = (Ax + B)\, e^x$

P.S.:

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x e^x & e^x \\ (x+1)e^x & e^x \end{vmatrix} = \left(x - (x+1)\right)e^{2x} = -e^{2x}$$

Example 4.3.3 (continued)

$$W_1 = \begin{vmatrix} 0 & y_2 \\ R & y_2' \end{vmatrix} = -y_2 R = -e^x e^x = -e^{2x}$$

$$\Rightarrow \quad u' = \frac{W_1}{W} = \frac{-e^{2x}}{-e^{2x}} = +1 \qquad \Rightarrow \quad u = x$$

$$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & R \end{vmatrix} = +y_1 R = x e^x e^x = x e^{2x}$$

$$\Rightarrow \quad v' = \frac{W_2}{W} = \frac{x e^{2x}}{-e^{2x}} = -x \qquad \Rightarrow \quad v = -\frac{x^2}{2}$$

$$y_P = u \cdot y_1 + v \cdot y_2 =$$

$$x\left(x e^x\right) + \left(-\frac{x^2}{2}\right)e^x = \frac{1}{2}x^2 e^x$$

G.S.:

$$y(x) = \underline{\underline{\left(\tfrac{1}{2}x^2 + Ax + B\right)e^x}}$$

Note:
Using the method of undetermined coefficients,

$R(x) = e^x$, but e^x and $x\,e^x$ are both in the complementary function.

Therefore the trial function for the particular solution is $y_P = c\,x^2\,e^x$.
Upon substituting this into the ODE, we find $c = 1/2$.

Modified Method of Undetermined Coefficients

If part of the complementary function, y_1, is included in the function $R(x)$, then try $y_P = f(x)\, y_1$ as a particular solution. Substitute into the ODE and solve for $f(x)$.

Example 4.3.3 (again)

$$y'' - 2y' + y = e^x$$

$$y_C = (Ax + B)\, e^x$$

Try $\quad y_P = f(x)\, e^x$

$\Rightarrow \quad y'_P = (f' + f)\, e^x$

$\Rightarrow \quad y''_P = (f'' + 2f' + f)\, e^x$

$\Rightarrow \quad y''_P - 2y'_P + y'_P = (f'' + 2f' + f - 2f' - 2f + f)\, e^x = e^x$

$\Rightarrow \quad f''(x) = 1$

$\Rightarrow \quad f'(x) = x$

$\Rightarrow \quad y_P = \frac{1}{2}x^2 e^x$

The general solution then follows,

$$y(x) = \underline{\underline{\left(\tfrac{1}{2}x^2 + Ax + B\right)e^x}}$$

Check on the general solution:

$$y'(x) = \left(\left(\tfrac{1}{2}x^2 + Ax + B\right) + (x + A)\right)e^x$$

$$y''(x) = \left(\left(\tfrac{1}{2}x^2 + Ax + B + x + A\right) + (x + A + 1)\right)e^x$$

$\Rightarrow \quad y'' - 2y' + y =$

$$\left\{\left(\tfrac{1}{2} - 2\left(\tfrac{1}{2}\right) + \tfrac{1}{2}\right)x^2 + \left((A+2) - 2(A+1) + A\right)x + \left((2A+B+1) - 2(A+B) + B\right)\right\}e^x$$

$$= (0 + 0 + 1)e^x = R(x)$$

✓

4.4 Higher Order Linear Ordinary Differential Equations

The n^{th} order ordinary differential equation

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + a_{n-2} \frac{d^2 y}{dx^2} + a_{n-1} \frac{dy}{dx} + a_n y = R(x)$$

can be solved as follows.

Form the auxiliary equation

$$\lambda^n + a_1 \lambda^{n-1} + \ldots + a_{n-2} \lambda^2 + a_{n-1} \lambda^1 + a_n = 0$$

Find all n values for λ.

Form the complementary function y_C, which will be a linear combination of

$\left\{ e^{\lambda_1 x}, e^{\lambda_2 x}, \ldots, e^{\lambda_n x} \right\}$ (except for repeated roots).

Complex conjugate pairs can be re-written in terms of sine and cosine functions.

Find a particular solution y_P (by inspection, undetermined coefficients, or variation of parameters, as extended to this higher order equation).

Write down the general solution $y = y_C + y_P$.

n initial and/or boundary conditions will be needed at this stage to evaluate all of the n arbitrary constants of integration.

Example 4.4.1 Find the general solution of

$$\frac{d^5 y}{dx^5} + 2 \frac{d^4 y}{dx^4} - 3 \frac{d^3 y}{dx^3} - 4 \frac{d^2 y}{dx^2} + 4 \frac{dy}{dx} = 8x$$

Auxiliary equation:

$$\lambda^5 + 2\lambda^4 - 3\lambda^3 - 4\lambda^2 + 4\lambda = 0$$

$$\Rightarrow \quad \lambda (\lambda^4 + 2\lambda^3 - 3\lambda^2 - 4\lambda + 4) = 0$$

$$\Rightarrow \quad \lambda (\lambda - 1) (\lambda^3 + 3\lambda^2 - 4) = 0$$

$$\Rightarrow \quad \lambda (\lambda - 1)^2 (\lambda^2 + 4\lambda + 4) = 0$$

$$\Rightarrow \quad \lambda (\lambda - 1)^2 (\lambda + 2)^2 = 0$$

$$\Rightarrow \quad \lambda = 0, 1, 1, -2, -2.$$

Complementary function:

$y_C = A + (Bx + C) e^x + (Dx + E) e^{-2x}$

Example 4.4.1 (continued)

Particular solution:

Cannot try $y_P = ax + b$ because a constant is included in the complementary function.

Therefore try $y_P = (ax + b)\, x = ax^2 + bx$

$\Rightarrow \quad y'_P = 2ax + b$

$\Rightarrow \quad y''_P = 2a$

$\Rightarrow \quad y'''_P = y^{(4)}{}_P = y^{(5)}{}_P = 0$

ODE $\Rightarrow \quad 0 + 0 - 0 - 8a + 8ax + 4b = 8x$

x^1: $\quad 8a = 8$

x^0: $\quad -8a + 4b = 0$

$\Rightarrow \quad a = 1\,,\ b = 2\,.$

Therefore the general solution is

$$y(x) = \underline{\underline{A + (Bx + C)e^{x} + (Dx + E)e^{-2x} + x^2 + 2x}}$$

Five initial conditions would be sufficient to evaluate the arbitrary constants A, B, C, D and E.

Also available:
Additional tutorial example of a second order ODE
(at "http://www.engr.mun.ca/~ggeorge/2422/notes/c4tutorl2.html")

[Cauchy-Euler ODEs will *not* be covered in this course.]

END OF CHAPTER 4