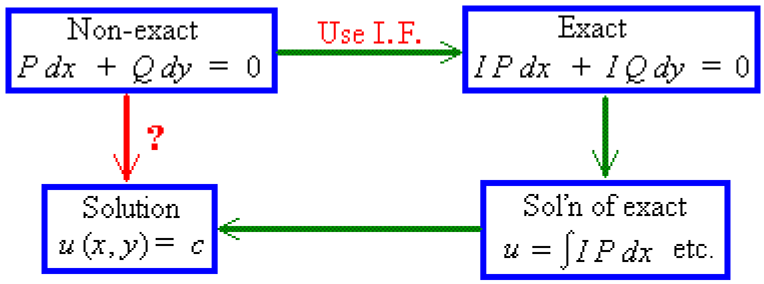
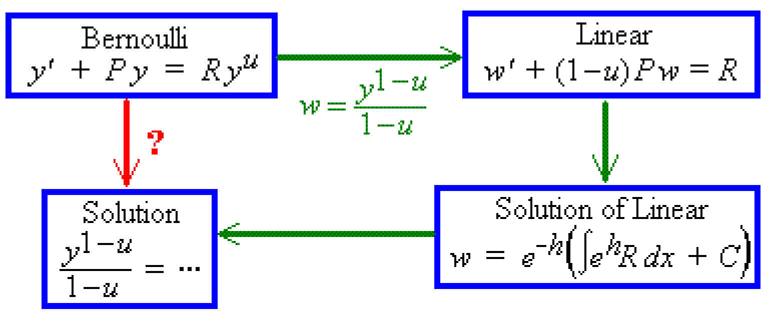
5.01 Transforms

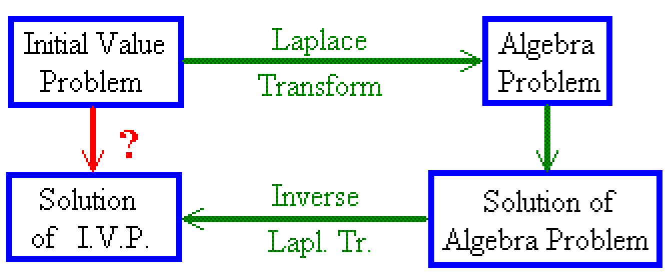
In some situations, a difficult problem can be transformed into an easier problem, whose solution can be transformed back into the solution of the original problem. For example, an integrating factor can sometimes be found to transform a non-exact first order first degree ordinary differential equation into an exact ODE:



One type of first order ODE [not considered in this semester] is the Bernoulli ODE, $y' + P\,y = R\,y^u$, which, upon a transformation of the dependent variable, becomes a linear ODE:



An **initial value problem** is an ordinary differential equation together with sufficient initial conditions to determine all of the arbitrary constants of integration. A Laplace transform will convert an initial value problem into a much easier algebra problem. The solution of the original problem is then the inverse Laplace transform of the solution to the algebra problem.



Uses of Laplace transforms include:

1) the solution of some ordinary differential equations
2) the solution of some integro-differential equations, such as

$$y(t) = g(t) + \int_0^t h(t-x)\,y(x)\,dx\ .$$

5.02 Some Laplace Transforms

If $f(t)$ is

- defined on $t > 0$,
- piece-wise continuous on $t > 0$
  (that is, only a finite number of finite discontinuities) and
- of exponential order
  [$|f(t)| < k\,e^{\alpha t}$ for all $t > 0$ and for some positive constants k and α],

then

$$F(s) = \lim_{m \to \infty} \int_0^m e^{-st} f(t)\,dt \text{ exists and}$$

the Laplace transform of $f(t)$ is

$$\boxed{F(s) = \int_0^\infty e^{-st} f(t)\,dt = \mathcal{L}\{f(t)\}}$$

Example 5.2.1

Find $\mathcal{L}\{\,1\,\}$.

$$F(s) = \int_0^\infty e^{-st}\,1\,dt = \left[\frac{e^{-st}}{-s}\right]_0^\infty = -\frac{1}{s}(0-1)$$

$$\therefore \quad \mathcal{L}\{1\} = \underline{\underline{\frac{1}{s}}}$$

Example 5.2.2

Find $\mathscr{L}\{\,t\,\}$.

$$F(s) = \int_0^\infty e^{-st}\, t\, dt$$

Integration by parts:

D		**I**
t		e^{-st}
	$+$	
1		$-\frac{1}{s}e^{-st}$
	$-$	
0		$+\frac{1}{s^2}e^{-st}$

$$\Rightarrow \quad F(s) = \left[\frac{-e^{-st}}{s^2}(st+1)\right]_0^\infty = \frac{-1}{s^2}(0-1)$$

$$\therefore \quad \mathscr{L}\{t\} = \underline{\underline{\frac{1}{s^2}}}$$

Linearity Property of Laplace Transforms

$\mathscr{L}\{\, a\, f(t) + b\, g(t)\,\}$ (where a, b are constants):

$$= \int_0^\infty e^{-st}\left(a\, f(t) + b\, g(t)\right)dt = a\int_0^\infty e^{-st} f(t)\, dt + b\int_0^\infty e^{-st} g(t)\, dt$$

$= a\,\mathscr{L}\{\, f(t)\,\} + b\,\mathscr{L}\{\, g(t)\,\}$, (provided all of the integrals exist).

[Thus the Laplace transform of a linear combination of functions is the linear combination of the separate Laplace transforms.]

Example 5.2.3

Find $\mathcal{L}\{2t-3\}$.

$$\mathcal{L}\{2t-3\} = 2\mathcal{L}\{t\} - 3\mathcal{L}\{1\} = 2\left(\frac{1}{s^2}\right) - 3\left(\frac{1}{s}\right)$$

$$\therefore \quad \mathcal{L}\{2t-3\} = \frac{2-3s}{s^2}$$

Example 5.2.4

Find $\mathcal{L}\{t^n\}$.

$$F(s) = \int_0^\infty e^{-st}\, t^n\, dt$$

D		**I**
t^n		e^{-st}
	+	
$n\,t^{n-1}$	— – ⟶ INTEGRATE	$-\frac{1}{s}e^{-st}$

$$\Rightarrow \quad F(s) = \left[t^n\left(-\frac{1}{s}e^{-st}\right)\right]_0^\infty + \frac{n}{s}\int_0^\infty e^{-st}t^{n-1}dt$$

$$\Rightarrow \quad \mathcal{L}\{t^n\} = (0-0) + \frac{n}{s}\mathcal{L}\{t^{n-1}\}$$ — a recurrence relation

$$\Rightarrow \quad \mathcal{L}\{t^n\} = \frac{n}{s}\cdot\frac{n-1}{s}\mathcal{L}\{t^{n-2}\} = \frac{n}{s}\cdot\frac{n-1}{s}\cdot\frac{n-2}{s}\mathcal{L}\{t^{n-3}\}$$

$$= \cdots = \frac{n}{s}\cdot\frac{n-1}{s}\cdot\frac{n-2}{s}\cdot\cdots\cdot\frac{3}{s}\cdot\frac{2}{s}\cdot\frac{1}{s}\mathcal{L}\{1\} = \frac{n!}{s^n}\cdot\frac{1}{s}$$

$$\therefore \quad \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}} \quad (n \in \mathbb{N})$$

Example 5.2.5

Find $\mathscr{L}\{ t^4 + 2t^3 - 3 \}$.

$$\mathscr{L}\left\{t^4 + 2t^3 - 3\right\} = \mathscr{L}\left\{t^4\right\} + 2\mathscr{L}\left\{t^3\right\} - 3\mathscr{L}\left\{1\right\}$$

$$= \frac{4!}{s^5} + 2\,\frac{3!}{s^4} - 3\,\frac{1}{s} = \underline{\underline{\frac{24}{s^5} + \frac{12}{s^4} - \frac{3}{s}}}$$

5.03 Laplace Transforms of Derivatives

$$\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\, dt = \left[e^{-st} f(t)\right]_0^\infty - \int_0^\infty -s\, e^{-st} f(t)\, dt \quad \text{(integration by parts)}$$

Therefore

$$\boxed{\mathcal{L}\{f'(t)\} = +s\,\mathcal{L}\{f(t)\} - f(0)},$$

(provided f is continuous at $t = 0$).

$$\mathcal{L}\{f''(t)\} = s\,\mathcal{L}\{f'(t)\} - f'(0) = s\left(s\,\mathcal{L}\{f(t)\} - f(0)\right) - f'(0)$$

Therefore

$$\boxed{\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - s\,f(0) - f'(0)},$$

(provided f and f' are continuous at $t = 0$).

Continuing this pattern, we can deduce the Laplace transform for any higher derivative of $f(t)$:

$$\boxed{\begin{aligned}\mathcal{L}\{f^{(n)}(t)\} &= s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) \\ &\quad - \cdots - s\,f^{(n-2)}(0) - f^{(n-1)}(0)\end{aligned}}$$

or

$$\boxed{\mathcal{L}\{f^{(n)}(t)\} = s^n\mathcal{L}\{f(t)\} - \sum_{i=0}^{n-1} s^{n-i-1} f^{(i)}(0)}$$

This result allows us to find the Laplace transform of an entire initial value problem.

Example 5.3.1

Find the Laplace transform of the initial value problem

$$y'' + y = t , \quad y(0) = 1 , \quad y'(0) = 0$$

and hence find the complete solution.

Let $Y(s) = \mathcal{L}\{ y(t) \}$, then

$$\mathcal{L}\{ y'' + y \} = s^2 Y - s\,y(0) - y'(0) + Y$$

$$\mathcal{L}\{ t \} = \frac{1}{s^2}$$

Therefore the Laplace transform of the initial value problem is

$$\left(s^2+1\right)Y - s - 0 = \frac{1}{s^2}$$

which is an algebra problem for $Y(s)$, (the Laplace transform of the complete solution). Solving the algebra problem:

$$Y(s) = \frac{1}{s^2+1}\left(s+\frac{1}{s^2}\right) = \frac{s}{s^2+1} + \frac{1}{s^2\left(s^2+1\right)} = \frac{s}{s^2+1} + \frac{1}{s^2} - \frac{1}{s^2+1}$$

In section 5.04 we shall see how to find the inverse Laplace transforms

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} = \cos t \quad \text{and} \quad \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t$$

The complete solution to the initial value problem is therefore

$$y(t) = \underline{\underline{\cos t + t - \sin t}}$$

5.04 First Shifting Theorem; Transforms of Exponential, Cosine and Sine Functions

$$\mathcal{L}\left\{e^{at} f(t)\right\} = \int_0^\infty e^{-st} e^{at} f(t)\,dt = \int_0^\infty e^{-(s-a)t} f(t)\,dt$$

The **first shifting theorem** then follows:

$$\text{If} \quad \mathcal{L}\left\{f(t)\right\} = F(s), \quad \text{then} \quad \mathcal{L}\left\{e^{at} f(t)\right\} = F(s-a)$$

Example 5.4.1

Find $\mathcal{L}\{t^3 e^{5t}\}$.

$$\mathcal{L}\left\{t^3\right\} = F(s) = \frac{3!}{s^{3+1}} = \frac{6}{s^4}$$

$$\Rightarrow \quad \mathcal{L}\left\{t^3 e^{5t}\right\} = F(s-5) = \frac{6}{(s-5)^4}$$ [using the first shift theorem]

Example 5.4.2

Find $\mathcal{L}\{e^{at}\}$.

$$\mathcal{L}\{1\} = F(s) = \frac{1}{s}$$

$$\Rightarrow \quad \mathcal{L}\left\{e^{at} \times 1\right\} = F(s-a) = \frac{1}{s-a}$$ [using the first shift theorem]

Example 5.4.3

Find $\mathcal{L}\{\cos \omega t\}$.

$$\cos \omega t = \frac{e^{j\omega t} + e^{-j\omega t}}{2}$$

But $\mathcal{L}\left\{e^{at}\right\} = \dfrac{1}{s-a}$

$$\therefore \quad \mathcal{L}\left\{\cos \omega t\right\} = \frac{1}{2}\left(\frac{1}{s-j\omega} + \frac{1}{s+j\omega}\right) = \frac{1}{2}\left(\frac{s+j\omega + s-j\omega}{s^2 - j^2\omega^2}\right) = \frac{2s}{2\left(s^2+\omega^2\right)}$$

Therefore

$$\boxed{\mathcal{L}\left\{\cos \omega t\right\} = \frac{s}{s^2+\omega^2}}$$

Example 5.4.4

Find $\mathcal{L}\{\sin \omega t\}$.

$$\sin \omega t = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}$$

But $\mathcal{L}\left\{e^{at}\right\} = \dfrac{1}{s-a}$

$$\therefore \quad \mathcal{L}\left\{\sin \omega t\right\} = \frac{1}{2j}\left(\frac{1}{s-j\omega} - \frac{1}{s+j\omega}\right) = \frac{1}{2j}\left(\frac{s+j\omega - s+j\omega}{s^2 - j^2\omega^2}\right) = \frac{1}{2j}\left(\frac{2j\omega}{s^2+\omega^2}\right)$$

Therefore

$$\boxed{\mathcal{L}\left\{\sin \omega t\right\} = \frac{\omega}{s^2+\omega^2}}$$

It then follows that

$$\boxed{\mathcal{L}^{-1}\left\{\frac{s}{s^2+\omega^2}\right\} = \cos \omega t} \quad \text{and} \quad \boxed{\mathcal{L}^{-1}\left\{\frac{1}{s^2+\omega^2}\right\} = \frac{\sin \omega t}{\omega}}$$

Example 5.4.5

Find $\mathcal{L}\{\sin \omega t\}$ directly, from the definition of a Laplace transform.

$$F(s) = \mathcal{L}\{\sin \omega t\} = \int_0^\infty e^{-st} \sin \omega t \, dt$$

D		**I**
e^{-st}		$\sin \omega t$
	$+$	
$-s\,e^{-st}$		$\dfrac{\cos \omega t}{-\omega}$
	$-$	
$+s^2 e^{-st}$	$— + \longrightarrow$ INTEGRATE	$\dfrac{\sin \omega t}{-\omega^2}$

$$\Rightarrow \quad F(s) = \left[\frac{-e^{-st}}{\omega^2} \left(\omega \cos \omega t + s \sin \omega t \right) \right]_0^\infty - \frac{s^2}{\omega^2} F(s)$$

$$\Rightarrow \quad \left(1 + \frac{s^2}{\omega^2} \right) F(s) = 0 - \left(\frac{-\omega}{\omega^2} + 0 \right)$$

$$\Rightarrow \quad \left(\frac{\omega^2 + s^2}{\omega^2} \right) F(s) = \frac{\omega}{\omega^2} \qquad \Rightarrow \quad F(s) = \frac{\omega}{s^2 + \omega^2}$$

Example 5.4.6

Find $\mathcal{L}\{\sin \omega t\}$, using $\mathcal{L}\{\cos \omega t\}$.

$$\sin \omega t = \frac{d}{dt} \left(\frac{\cos \omega t}{-\omega} \right). \qquad \text{Let} \quad f(t) = \frac{\cos \omega t}{-\omega}$$

$$\mathcal{L}\{ f'(t) \} = s\,\mathcal{L}\{ f(t) \} - f(0)$$

$$\Rightarrow \quad \mathcal{L}\{\sin \omega t\} = s\,\mathcal{L}\left\{ \frac{\cos \omega t}{-\omega} \right\} - \left(\frac{\cos 0}{-\omega} \right)$$

$$= -\frac{s}{\omega} \cdot \frac{s}{s^2 + \omega^2} + \frac{1}{\omega} = \frac{-s^2 + \left(s^2 + \omega^2 \right)}{\omega \left(s^2 + \omega^2 \right)} = \frac{\omega^2}{\omega \left(s^2 + \omega^2 \right)}$$

$$\therefore \quad \mathcal{L}\{\sin \omega t\} = \frac{\omega}{s^2 + \omega^2}$$

Example 5.4.7

Find $\mathcal{L}\{\, e^{at} \sin \omega t \,\}$.

$$\mathcal{L}\left\{\sin \omega t\right\} = \frac{\omega}{s^2 + \omega^2}$$

Using the first shift theorem, we have, immediately,

$$\boxed{\mathcal{L}\left\{e^{at} \sin \omega t\right\} = \frac{\omega}{(s-a)^2 + \omega^2}}$$

Similarly,

$$\boxed{\mathcal{L}\left\{e^{at} \cos \omega t\right\} = \frac{s-a}{(s-a)^2 + \omega^2}}$$

Example 5.4.8

Find $\mathcal{L}\{\, \tan \omega t \,\}$.

$\tan \omega t$ is not of exponential order.
[Also, it has infinitely many infinite discontinuities.]
Therefore

$\mathcal{L}\{\, \tan \omega t \,\}$ does not exist.

Example 5.4.9

Find $\mathcal{L}\{\, \sinh at \,\}$.

$$\sinh at = \frac{e^{at} - e^{-at}}{2}$$

$$\Rightarrow \quad \mathcal{L}\left\{\sinh at\right\} = \frac{1}{2}\left(\frac{1}{s-a} - \frac{1}{s+a}\right) = \frac{1}{2}\left(\frac{s+a \;-\; s+a}{s^2 - a^2}\right) = \underline{\underline{\frac{a}{s^2 - a^2}}}$$

Similarly, $\mathcal{L}\left\{\cosh at\right\} = \underline{\underline{\dfrac{s}{s^2 - a^2}}}$

Summary so far:

Definition:

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

Linearity:

$$\mathcal{L}\{a f(t) + b g(t)\} = a\,\mathcal{L}\{f(t)\} + b\,\mathcal{L}\{g(t)\} \qquad (a, b = \text{constants})$$

Polynomial functions:

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}} \quad \Rightarrow \quad \mathcal{L}^{-1}\left\{\frac{1}{s^n}\right\} = \frac{t^{n-1}}{(n-1)!}$$

First Shift Theorem:

$$\mathcal{L}\{f(t)\} = F(s) \;\Rightarrow\; \mathcal{L}\{e^{at} f(t)\} = F(s-a) \;\text{ and }\; \mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$$

Trigonometric Functions:

$$\mathcal{L}\{e^{at}\sin \omega t\} = \frac{\omega}{(s-a)^2 + \omega^2} \quad \Rightarrow \quad \mathcal{L}^{-1}\left\{\frac{1}{(s-a)^2 + \omega^2}\right\} = \frac{e^{at}\sin \omega t}{\omega}$$

$$\mathcal{L}\{e^{at}\cos \omega t\} = \frac{s-a}{(s-a)^2 + \omega^2} \quad \Rightarrow \quad \mathcal{L}^{-1}\left\{\frac{s-a}{(s-a)^2 + \omega^2}\right\} = e^{at}\cos \omega t$$

Hyperbolic Functions:

$$\mathcal{L}\{e^{at}\sinh bt\} = \frac{b}{(s-a)^2 - b^2} \quad \Rightarrow \quad \mathcal{L}^{-1}\left\{\frac{1}{(s-a)^2 - b^2}\right\} = \frac{e^{at}\sinh bt}{b}$$

$$\mathcal{L}\{e^{at}\cosh bt\} = \frac{s-a}{(s-a)^2 - b^2} \quad \Rightarrow \quad \mathcal{L}^{-1}\left\{\frac{s-a}{(s-a)^2 - b^2}\right\} = e^{at}\cosh bt$$

Derivatives:

$$\mathcal{L}\{f'(t)\} = s\,\mathcal{L}\{f(t)\} - f(0)$$

$$\mathcal{L}\{f''(t)\} = s^2\,\mathcal{L}\{f(t)\} - s\,f(0) - f'(0)$$

etc.

When trying to find the inverse Laplace transform of a rational function $F(s)$, one usually attempts to break the function up into its partial fractions, with real linear and quadratic denominators, so that one may read the inverses from the table on this page.

5.05 Applications to Initial Value Problems

Example 5.5.1

Solve the initial value problem

$$y'' - 5y' + 6y = 0 , \quad y(0) = 1, \; y'(0) = 0$$

Let $Y(s) = \mathcal{L}\{ y(t) \}$.

$$\mathcal{L}\{ y'' \} = s^2 Y(s) - s\,y(0) - y'(0) = s^2 Y(s) - s$$

$$\mathcal{L}\{ y' \} = s\,Y(s) - y(0) = s\,Y(s) - 1$$

Taking the Laplace transform of the entire initial value problem:

$$\mathcal{L}\{ y'' - 5y' + 6y \} = \mathcal{L}\{ 0 \} \quad \Rightarrow$$

$$\left(s^2 - 5s + 6\right) Y - s + 5 = 0$$

$$\Rightarrow \quad Y = \frac{s-5}{s^2 - 5s + 6} = \frac{s-5}{(s-2)(s-3)}$$

$$= \frac{\left(\frac{2-5}{2-3}\right)}{s-2} + \frac{\left(\frac{3-5}{3-2}\right)}{s-3} \qquad \text{(cover-up rule)}$$

$$\Rightarrow \quad Y(s) = \frac{3}{s-2} - \frac{2}{s-3}$$

But $\mathcal{L}\left\{ e^{at} \right\} = \dfrac{1}{s-a}$

$$\therefore \quad y(t) = \underline{\underline{3\,e^{2t} - 2\,e^{3t}}}$$

Check 1:

$$y' = 6\,e^{2t} - 6\,e^{3t} \quad \Rightarrow \quad y'' = 12\,e^{2t} - 18\,e^{3t}$$

$$\Rightarrow \quad y'' - 5y' + 6y = (12 - 30 + 18)\,e^{2t} + (-18 + 30 - 12)\,e^{3t} = 0 \quad \checkmark$$

and $y(0) = 3 - 2 = 1$ ✓ and $y'(0) = 6 - 6 = 0$ ✓

Example 5.5.1 (continued)

Check 2:

Solution by Chapter 4 method:

$$y'' - 5y' + 6y = 0$$

A.E.: $\lambda^2 - 5\lambda + 6 = 0$

$\Rightarrow \quad (\lambda - 2)(\lambda - 3) = 0 \quad \Rightarrow \quad \lambda = 2, 3.$

C.F.: $y_C = A\,e^{2t} + B\,e^{3t}$

P.S.: $y_P = 0$

G.S. = C.F.

$$y'(t) = 2A\,e^{2t} + 3B\,e^{3t}$$

I.C.: $y(0) = A + B = 1 \quad \Rightarrow \quad B = 1 - A$

$$y'(0) = 2A + 3B = 2A + 3(1 - A) = 0$$

$\Rightarrow \quad 3 = (3 - 2)A \quad \Rightarrow \quad A = 3$

$\Rightarrow \quad B = 1 - 3 = -2$

Therefore the complete solution is

$$y(t) = \underline{\underline{3\,e^{2t} - 2\,e^{3t}}} \quad \checkmark$$

Example 5.5.2

Solve the initial value problem

$$y'' + 4y' + 13y = 26\, e^{-4t}, \quad y(0) = 5, \quad y'(0) = -29$$

Let $Y(s) = \mathcal{L}\{\, y(t)\,\}$.

$$\mathcal{L}\{\, y''\,\} = s^2\, Y(s) - s\, y(0) - y'(0) = s^2\, Y(s) - 5s + 29$$

$$\mathcal{L}\{\, y'\,\} = s\, Y(s) - y(0) = s\, Y(s) - 5$$

$$\mathcal{L}\left\{26\, e^{-4t}\right\} = \frac{26}{s+4}$$

Taking the Laplace transform of the entire initial value problem:

$$\mathcal{L}\{\, y'' + 4y' + 13y\,\} = \mathcal{L}\left\{26\, e^{-4t}\right\} \quad \Rightarrow$$

$$\left(s^2 + 4s + 13\right)Y - 5s + 29 - 20 = \frac{26}{s+4}$$

$$\Rightarrow \quad \left(s^2 + 4s + 13\right)Y = 5s - 9 + \frac{26}{s+4} = \frac{5s^2 + 11s - 10}{s+4}$$

$$\Rightarrow \quad Y(s) = \frac{5s^2 + 11s - 10}{(s+4)\left(s^2 + 4s + 13\right)}$$

Recall the three standard inverse Laplace transforms:

$$\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$$

$$\mathcal{L}^{-1}\left\{\frac{s-a}{(s-a)^2 + \omega^2}\right\} = e^{at}\cos\omega t$$

$$\mathcal{L}^{-1}\left\{\frac{\omega}{(s-a)^2 + \omega^2}\right\} = e^{at}\sin\omega t$$

<u>Example 5.5.2</u> (continued)

We need to express $Y(s)$ in partial fractions and to complete the square in the quadratic denominator.

$$Y(s) = \frac{5s^2+11s-10}{(s+4)(s^2+4s+13)} = \frac{a}{s+4} + \frac{b(s+2)+c(3)}{(s+2)^2+3^2}$$

This rearrangement allows us to see immediately that the solution is
$y(t) = \mathscr{L}^{-1}\{Y(s)\} = a e^{-4t} + b e^{-2t}\cos 3t + c e^{-2t}\sin 3t$
We need to find the values of the numerators a, b and c:

Cover-up rule for a:

$$a = \frac{5(-4)^2+11(-4)-10}{\boxed{\times}\left((-4)^2+4(-4)+13\right)} = \frac{26}{13} = 2$$

$$\Rightarrow \quad 5s^2+11s-10 = 2(s^2+4s+13) + b(s+2)(s+4) + 3c(s+4)$$

$s = -2$: $\quad 20-22-10 = 2(4-8+13) + 0 + 3c(2)$

$\Rightarrow \quad -12-18 = 6c \quad \Rightarrow \quad 6c = -30 \quad \Rightarrow \quad c = -5$

$s = 0$: $\quad -10 = 2(13) + 8b + 12(-5)$

$\Rightarrow \quad 8b = -10-26+60 = 24 \quad \Rightarrow \quad b = 3$

Therefore

$$Y(s) = \frac{2}{s+4} + \frac{3(s+2)-5(3)}{(s+2)^2+3^2}$$

$\Rightarrow$

$$\underline{\underline{y(t) = 2e^{-4t} + e^{-2t}(3\cos 3t - 5\sin 3t)}}$$

Example 5.5.2 (by a Chapter 4 method)

$$y'' + 4y' + 13y = 26\,e^{-4t}, \quad y(0) = 5, \quad y'(0) = -29$$

A.E.: $\lambda^2 + 4\lambda + 13 = 0$

$$\Rightarrow \quad \lambda = \frac{-4 \pm \sqrt{16-52}}{2} = -2 \pm 3j$$

C.F.: $y_C = e^{-2t}(A\cos 3t + B\sin 3t)$

P.S.: Try $y_P = d\,e^{-4t}$ [method of undetermined coefficients]

$y''_P + 4y'_P + 13y_P = d\,e^{-4t}(16 - 16 + 13) = 26\,e^{-4t}$

$\Rightarrow \quad d = 2$

G.S.: $y = y_C + y_P = e^{-2t}(A\cos 3t + B\sin 3t) + 2\,e^{-4t}$

$\Rightarrow \quad y' = e^{-2t}(-3A\sin 3t + 3B\cos 3t - 2A\cos 3t - 2B\sin 3t) - 8\,e^{-4t}$

I.C.: $y(0) = A + 2 = 5 \quad \Rightarrow \quad A = 3$

$y'(0) = 0 + 3B - 2(3) - 0 - 8 = -29 \quad \Rightarrow \quad 3B = -15 \quad \Rightarrow \quad B = -5$

Complete solution:

$$y(t) = \underline{\underline{2e^{-4t} + e^{-2t}\left(3\cos 3t - 5\sin 3t\right)}}$$

5.06 Laplace Transform of an Integral

Let $f(t) = \int_0^t g(\tau)\,d\tau$, then

$f(0) = 0$,

$g(t) = f'(t)$ and

$\mathcal{L}\{f'(t)\} = s\,\mathcal{L}\{f(t)\} - \cancel{f(0)} \quad \Rightarrow$

$$\boxed{\mathcal{L}\left\{\int_0^t g(\tau)\,d\tau\right\} = \frac{1}{s}\mathcal{L}\{g(t)\}}$$

[Division in the s domain corresponds to integration in the t domain.
If the initial conditions are all zero, then
Multiplication in the s domain corresponds to differentiation in the t domain.]

Also

$$\boxed{\mathcal{L}^{-1}\left\{\frac{1}{s}G(s)\right\} = \int_0^t \mathcal{L}^{-1}\{G(s)\}\,d\tau}$$

Example 5.6.1

Find the function $f(t)$ whose Laplace transform is $F(s) = \dfrac{1}{s(s^2+\omega^2)}$.

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+\omega^2}\right\} = \frac{\sin\omega t}{\omega}$$

$$\Rightarrow \quad f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s}\cdot\frac{1}{s^2+\omega^2}\right\} = \int_0^t \mathcal{L}^{-1}\left\{\frac{1}{s^2+\omega^2}\right\}d\tau = \int_0^t \frac{\sin\omega\tau}{\omega}\,d\tau$$

$$= \frac{1}{\omega}\left[-\frac{\cos\omega\tau}{\omega}\right]_0^t = \frac{1-\cos\omega t}{\omega^2}$$

Therefore

$$\boxed{\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+\omega^2)}\right\} = \frac{1-\cos\omega t}{\omega^2}}$$

<u>Example 5.6.1</u> (Alternative solution, using partial fractions)

$$\frac{1}{s\left(s^2+\omega^2\right)} = \frac{\left(\frac{1}{\omega^2}\right)}{s} + \frac{\left(\frac{-s}{\omega^2}\right)}{s^2+\omega^2}$$

$$\Rightarrow \quad f(t) = \frac{1}{\omega^2}\left(\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} - \mathscr{L}^{-1}\left\{\frac{s}{s^2+\omega^2}\right\}\right)$$

$$= \frac{1}{\omega^2}(1 - \cos\omega t)$$

<u>Example 5.6.2</u>

Find the function $f(t)$ whose Laplace transform is $F(s) = \dfrac{1}{s^2\left(s^2+\omega^2\right)}$.

$$f(t) = \mathscr{L}^{-1}\left\{\frac{1}{s}\cdot\frac{1}{s\left(s^2+\omega^2\right)}\right\} = \int_0^t \mathscr{L}^{-1}\left\{\frac{1}{s\left(s^2+\omega^2\right)}\right\}d\tau$$

$$= \int_0^t \frac{1 - \cos\omega\tau}{\omega^2}\,d\tau = \frac{1}{\omega^2}\left[\tau - \frac{\sin\omega\tau}{\omega}\right]_0^t$$

Therefore

$$\boxed{\mathscr{L}^{-1}\left\{\frac{1}{s^2\left(s^2+\omega^2\right)}\right\} = \frac{1}{\omega^3}(\omega t - \sin\omega t)}$$

Alternative method, using partial fractions:

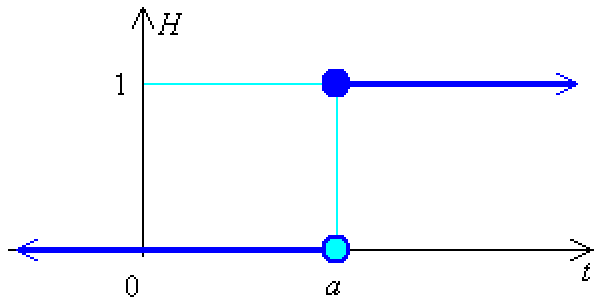
$$\frac{1}{s^2\left(s^2+\omega^2\right)} = \frac{\left(\frac{1}{\omega^2}\right)}{s^2} + \frac{\left(\frac{-1}{\omega^2}\right)}{s^2+\omega^2}$$

$$\Rightarrow \quad f(t) = \frac{1}{\omega^2}\left(t - \frac{\sin\omega t}{\omega}\right)$$

5.07 Heaviside Function; Second Shift Theorem

The **Heaviside function** $H(t-a)$ (also known as the **unit step function** $u(t-a)$) is defined by

$$H(t-a) = \begin{cases} 0 & (t<a) \\ 1 & (t \ge a) \end{cases} \qquad (a \ge 0)$$



The Laplace transform of the Heaviside function is:

$$F(s) = \mathcal{L}\{H(t-a)\} = \int_0^\infty e^{-st} H(t-a)\,dt$$

$$= \int_0^a e^{-st} 0\,dt + \int_a^\infty e^{-st} 1\,dt$$

$$= 0 + \left[\frac{e^{-st}}{-s}\right]_a^\infty = -\frac{1}{s}\left(0 - e^{-as}\right)$$

Therefore

$$\boxed{\mathcal{L}\{H(t-a)\} = \frac{e^{-as}}{s}}$$

If

$$f(t) = \begin{cases} g(t) & (0 \le t < a) \\ h(t) & (t \ge a) \end{cases},$$

then

$$f(t) = g(t) + \left(\underset{\substack{\uparrow \\ ON}}{h(t)} - \underset{\substack{\uparrow \\ OFF}}{g(t)} \right) H(t-a)$$

[The function $h(t)$ is "switched on" and $g(t)$ is "switched off" at time $t = a$.]

Example 5.7.1 (This is part of a question on Problem Set 9)

Express the function

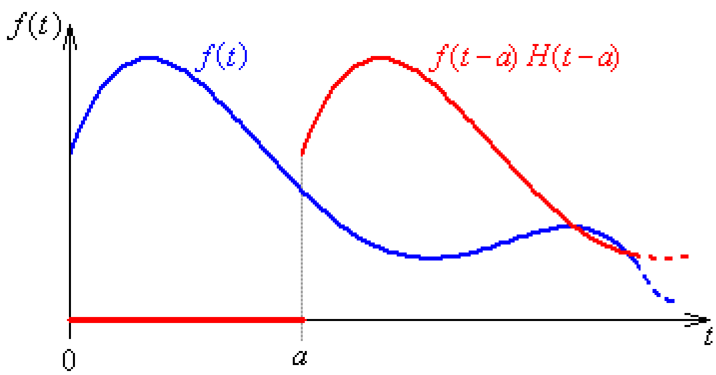
$$f(x) = \begin{cases} 2x & (x<2) \\ 4 & (x \ge 2) \end{cases}$$

in a single line definition.

Answer: $f(x) = \underline{2x + (4-2x)\,H(x-2)}$

The Second Shift Theorem

The result of shifting the graph $y = f(t)$ (defined only for $t > 0$) a units to the right, is the graph $y = f(t-a)\,H(t-a)$:



The Laplace transform of the shifted function of t is:

$$\mathcal{L}\{H(t-a)f(t-a)\} = \int_0^\infty e^{-st}H(t-a)f(t-a)\,dt$$

$$= 0 + \int_a^\infty e^{-st}f(t-a)\,dt$$

Let $u = t-a \quad \Rightarrow \quad du = dt$ and

$$\mathcal{L}\{H(t-a)f(t-a)\} = \int_0^\infty e^{-s(u+a)}f(u)\,du = e^{-as}\int_0^\infty e^{-su}f(u)\,du$$

The second shift theorem (for $a > 0$) is

$$\boxed{\mathcal{L}\{H(t-a)f(t-a)\} = e^{-as}\mathcal{L}\{f(t)\}}$$

Another way to express the second shift theorem is:

$$\mathcal{L}^{-1}\{F(s)\} = f(t) \quad \Rightarrow \quad \mathcal{L}^{-1}\{e^{-as}F(s)\} = H(t-a)\,f(t-a)$$

Example 5.7.2

Find $\mathcal{L}\{\, H(t-4)\,(t-4)^3 \,\}$.

$\mathcal{L}\{\, H(t-4)\,(t-4)^3 \,\} = e^{-4s}\ \mathcal{L}\{\, t^3 \,\}$ (by the second shift theorem)

$$= e^{-4s}\frac{3!}{s^4} = \underline{\underline{\frac{6\,e^{-4s}}{s^4}}}$$

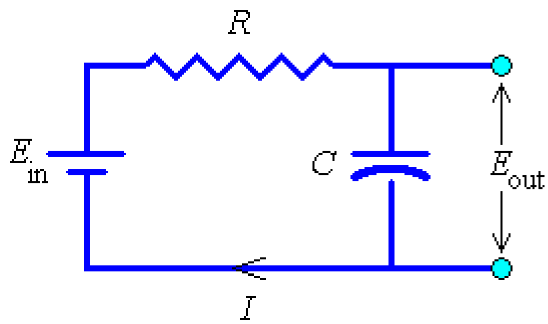
Example 5.7.3

Find the function whose Laplace transform is $\dfrac{e^{-5s}}{s^2+4}$.

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+2^2}\right\} = \frac{\sin 2t}{2} \quad \Rightarrow \quad \mathcal{L}^{-1}\left\{\frac{e^{-5s}}{s^2+4}\right\} = \underline{\underline{\frac{1}{2}\sin 2(t-5)\,H(t-5)}}$$

Example 5.7.4

In the RC circuit shown here, there is no charge on the capacitor and no current flowing at the time $t = 0$.

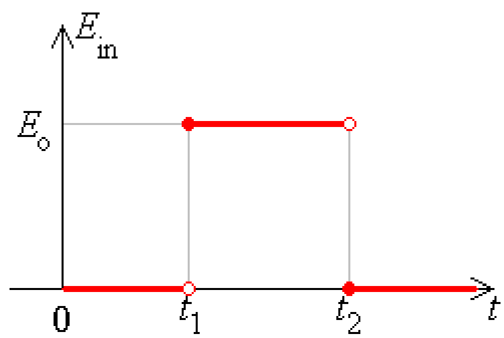


The input voltage E_{in} is a constant E_o during the time $t_1 < t < t_2$ and is zero at all other times.

Find the output voltage E_{out} for this circuit.

Initial conditions:

$q(0) = i(0) = 0 \qquad \left[\text{Note: } \dfrac{dq}{dt} = i\right]$



$$E_{\text{in}} = \begin{cases} E_o & (t_1 < t < t_2) \\ 0 & (\text{otherwise}) \end{cases}$$

$$\Rightarrow \quad E_{\text{in}} = E_o \left(H(t - t_1) - H(t - t_2) \right)$$

Equating voltage drops around the circuit with E_{in} :

$$R\,i + \frac{q}{C} = E_{\text{in}}$$

$$\Rightarrow \quad R\,q' + \frac{q}{C} = E_o\left(H(t-t_1) - H(t-t_2)\right)$$

Let $Q(s) = \mathcal{L}\{\, q(t)\,\}$. Taking the Laplace transform of this initial value problem:

$$R(sQ - 0) + \frac{1}{C}Q = E_o\left(\frac{e^{-t_1 s}}{s} - \frac{e^{-t_2 s}}{s}\right)$$

$$\Rightarrow \quad Q(s) = \frac{E_o}{s\left(Rs + \dfrac{1}{C}\right)}\left(e^{-t_1 s} - e^{-t_2 s}\right)$$

Example 5.7.4 (continued)

Using the cover-up rule for the partial fractions:

$$\frac{1}{Rs\left(s+\frac{1}{RC}\right)} = \frac{1}{\not R}\left(\frac{\not R C}{s} + \frac{-\not R C}{s+\frac{1}{RC}}\right)$$

$$\Rightarrow \quad \mathcal{L}^{-1}\left\{\frac{E_o}{s\left(Rs+\frac{1}{C}\right)}\right\} = E_o C\left(1 - e^{-t/(RC)}\right)$$

Therefore, using the second shift theorem,

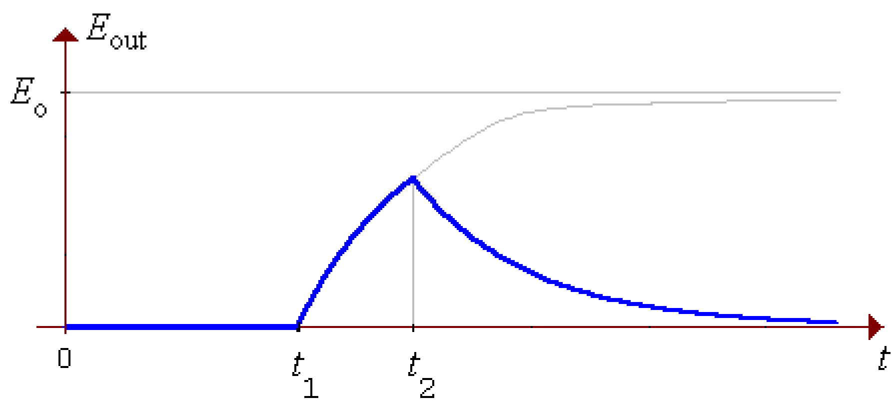
$$\mathcal{L}^{-1}\left\{\frac{E_o e^{-t_1 s}}{s\left(Rs+\frac{1}{C}\right)}\right\} = E_o C\left(1 - e^{-(t-t_1)/(RC)}\right)H(t-t_1)$$

$$\Rightarrow \quad q(t) = E_o C\left\{\left(1 - e^{-(t-t_1)/(RC)}\right)H(t-t_1) - \left(1 - e^{-(t-t_2)/(RC)}\right)H(t-t_2)\right\}$$

$$E_{\text{out}} = \frac{q}{C} \quad \Rightarrow$$

$$E_{\text{out}} = E_o\left\{\left(1 - e^{-\left(\frac{t-t_1}{RC}\right)}\right)H(t-t_1) - \left(1 - e^{-\left(\frac{t-t_2}{RC}\right)}\right)H(t-t_2)\right\}$$

Graph of the output voltage against time:



The capacitor charges from 0 at time t_1 then discharges from time t_2 onwards.

Example 5.7.5

Find the complete solution of the initial value problem

$$\frac{d^2y}{dt^2} + 4y = f(t) = \begin{cases} 0 & (0 \le t < 3) \\ t & (t \ge 3) \end{cases} \quad ; \qquad y(0) = y'(0) = 0 .$$

Let $Y(s) = \mathscr{L}\{ y(t) \}$

$f(t) = t\,H(t-3) = (t-3)\,H(t-3) + 3\,H(t-3)$

$\Rightarrow \quad \mathscr{L}\{ f(t) \} = e^{-3s}\,\mathscr{L}\{ t + 3 \}$ [second shift theorem]

$\mathscr{L}\{ y'' + 4y \} = \mathscr{L}\{ f(t) \}$

$$\Rightarrow \quad s^2 Y + 4Y = \frac{e^{-3s}}{s^2} + \frac{3e^{-3s}}{s}$$

$$\Rightarrow \quad Y(s) = e^{-3s}\left(\frac{3s+1}{s^2\left(s^2+4\right)}\right)$$

But $\dfrac{3s+1}{s^2\left(s^2+4\right)} = \dfrac{a}{s} + \dfrac{b}{s^2} + \dfrac{cs + d(2)}{s^2+4}$

$\Rightarrow \quad 3s + 1 = as(s^2+4) + b(s^2+4) + c\,s^3 + 2d\,s^2$

Matching coefficients of s^n:

s^0: $1 = 0 + 4b + 0 + 0 \Rightarrow b = 1/4$

s^1: $3 = 4a + 0 + 0 + 0 \Rightarrow a = 3/4$

s^2: $0 = 0 + 1/4 + 0 + 2d \Rightarrow d = -1/8$

s^3: $0 = 3/4 + 0 + c + 0 \Rightarrow c = -3/4$

$$\Rightarrow \quad \frac{3s+1}{s^2\left(s^2+4\right)} = \frac{3}{4}\left(\frac{1}{s}\right) + \frac{1}{4}\left(\frac{1}{s^2}\right) - \frac{3}{4}\left(\frac{s}{s^2+4}\right) - \frac{1}{8}\left(\frac{2}{s^2+4}\right)$$

<u>Example 5.7.5</u> (continued)

$$\Rightarrow \quad \mathcal{L}^{-1}\left\{ \frac{3s+1}{s^2\left(s^2+4\right)} \right\} = \frac{3}{4} + \frac{1}{4}t - \frac{3}{4}\cos 2t - \frac{1}{8}\sin 2t$$

One version of the second shift theorem states

$$\mathcal{L}^{-1}\left\{ F(s) \right\} = f(t) \quad \Rightarrow \quad \mathcal{L}^{-1}\left\{ e^{-as}F(s) \right\} = f(t-a)H(t-a)$$

Therefore

$$y(t) = \left(\frac{3}{4} + \frac{t-3}{4} - \frac{3}{4}\cos 2(t-3) - \frac{1}{8}\sin 2(t-3) \right) H(t-3)$$

or equivalently

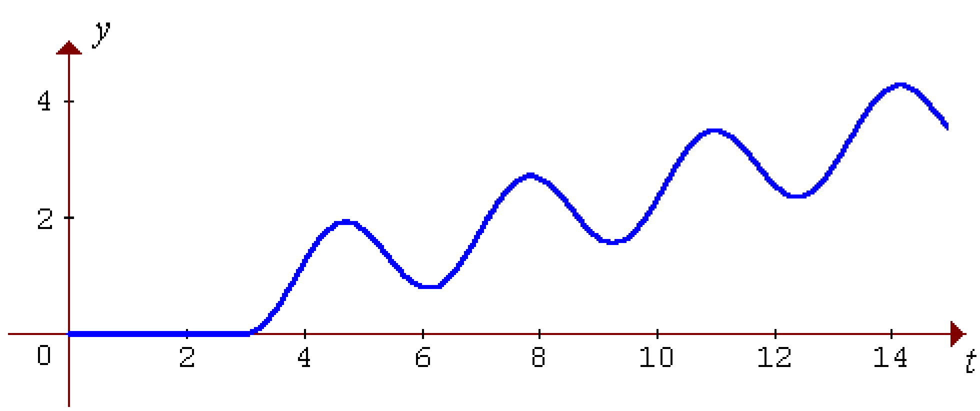
$$y(t) = \begin{cases} 0 & (0 \le t < 3) \\ \frac{1}{8}\left(2t - 6\cos 2(t-3) - \sin 2(t-3)\right) & (t \ge 3) \end{cases}$$

[Note: an alternative to partial fractions is:

$$\mathcal{L}^{-1}\left\{ \frac{3s+1}{s^2\left(s^2+4\right)} \right\} = 3\,\mathcal{L}^{-1}\left\{ \frac{1}{s\left(s^2+4\right)} \right\} + \mathcal{L}^{-1}\left\{ \frac{1}{s^2\left(s^2+4\right)} \right\}$$

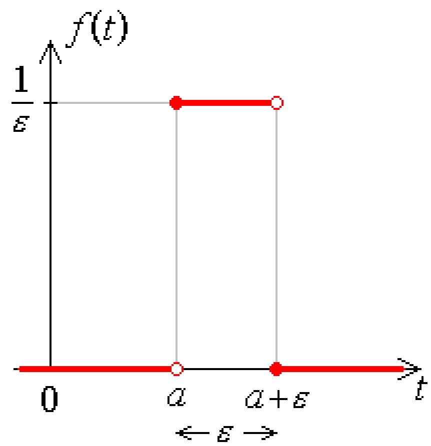
Quoting standard inverses, we obtain immediately

$$\mathcal{L}^{-1}\left\{ \frac{3s+1}{s^2\left(s^2+4\right)} \right\} = 3\left(\frac{1 - \cos 2t}{2^2} \right) + \left(\frac{2t - \sin 2t}{2^3} \right) = \frac{3 - 3\cos 2t}{4} + \frac{2t - \sin 2t}{8} \quad]$$



5.08 The Dirac Delta Function

Let $f(t) = \begin{cases} \dfrac{1}{\varepsilon} & (a \le t < a+\varepsilon) \\ 0 & (\text{otherwise}) \end{cases}$



Area under graph = Area of rectangle

$$\int_0^\infty f(t)\,dt = \frac{1}{\varepsilon}\times\varepsilon = 1$$

Also

$$f(t) = \frac{1}{\varepsilon}\big(H(t-a) - H(t-a-\varepsilon)\big)$$

Define the Dirac delta function to be

$$\delta(t-a) = \lim_{\varepsilon\to 0^+} f(t)$$

then

$$\mathcal{L}\{f(t)\} = \frac{1}{\varepsilon}\left(\frac{e^{-as}}{s} - \frac{e^{-(a+\varepsilon)s}}{s}\right)$$

$$\Rightarrow \quad \mathcal{L}\{\delta(t-a)\} = \frac{e^{-as}}{s}\lim_{\varepsilon\to 0^+}\left(\frac{1 - e^{-\varepsilon s}}{\varepsilon}\right)$$

$$\overset{\text{H}}{=} \frac{e^{-as}}{s}\lim_{\varepsilon\to 0^+}\left(\frac{0 + s\,e^{-\varepsilon s}}{1}\right) = \frac{e^{-as}}{s}\,\frac{s}{1} = e^{-as}$$

Therefore (for $a \ge 0$)

$$\boxed{\mathcal{L}\{\delta(t-a)\} = e^{-as}}$$

and $\mathcal{L}\{\delta(t)\} = 1$.

Also, for $a > 0$, the total area under the graph is 1 (even in the limit as $\varepsilon \to 0^+$), so

$$\boxed{\int_0^\infty \delta(t-a)\,dt = 1}$$

For $g(t)$ = any function that is continuous everywhere on $[0, \infty)$,

$$\int_0^\infty g(t)\delta(t-a)\,dt = \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon}\int_a^{a+\varepsilon} g(t)\,dt$$

$$= \lim_{\varepsilon \to 0^+} g(t_o), \quad \text{where} \quad a < t_o < a+\varepsilon \qquad [\text{mean value theorem}]$$

$$= g(a)$$

Therefore

$$\boxed{\int_0^\infty g(t)\delta(t-a)\,dt = g(a)}$$

(which is the sifting property of the Dirac delta function).

$$\int_c^d f(t)\delta(t-a)\,dt = \begin{cases} f(a) & (\text{if } \ c<a<d) \\ 0 & (a<c \ \text{ or } \ a>d) \end{cases}$$

<u>Example 5.8.1</u>

A damped mass-spring system, (with damping constant = $c = 3m$ and spring modulus = $k = 2m$, where m is the mass), is at rest in its equilibrium position, until an impulse of 1 Ns is applied at time $t = 1$ s. Find the response $y(t)$.

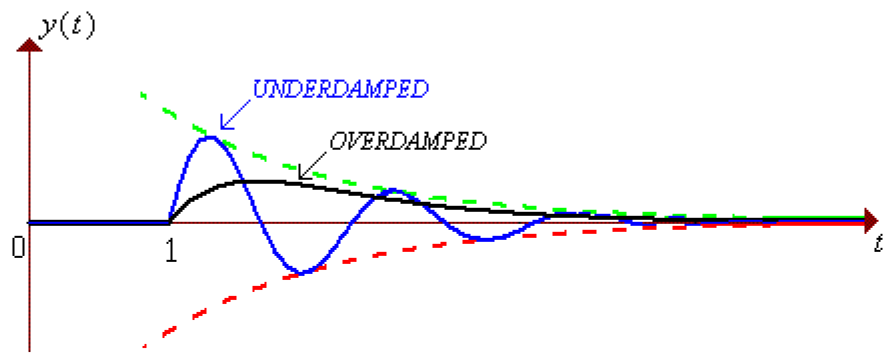
ODE: $\quad y'' + 3y' + 2y = \delta(t-1)$

I.C.: $\quad y'(0) = y(0) = 0$

One can anticipate some features of the response. The forcing function is zero until $t = 1$. Thus there is nothing to disturb the system until that instant. We can deduce that $y(t) \equiv 0$ during $t < 1$. The mass-spring system is set into motion abruptly by the arrival of the impulse at the instant $t = 1$, but the forcing term is again zero thereafter. The damping force will asymptotically restore the system to its equilibrium state. The only question remaining is whether the system is under-damped or not (see the diagram on the next page).

Example 5.8.1 (continued)

Anticipated response:



The auxiliary equation is $\lambda^2 + 3\lambda + 2 = 0$. The discriminant is

$D = B^2 - 4C = 3^2 - 4(2) = 9 - 8 > 0$

$\Rightarrow$ system is over-damped.

Solution, using Laplace transforms:

$$y'' + 3y' + 2y = \delta(t-1)$$

$$y'(0) = y(0) = 0$$

Let $Y(s) = \mathcal{L}\{ y(t) \}$

$$\Rightarrow \quad s^2 Y + 3sY + 2Y = e^{-1s}$$

$$\Rightarrow \quad Y(s) = \frac{e^{-s}}{s^2 + 3s + 2} = \frac{e^{-s}}{(s+1)(s+2)}$$

But $\dfrac{1}{(s+1)(s+2)} = \dfrac{\left(\frac{1}{-1+2}\right)}{s+1} + \dfrac{\left(\frac{1}{-2+1}\right)}{s+2} = \dfrac{1}{s+1} - \dfrac{1}{s+2}$ [cover-up rule]

<u>Example 5.8.1</u> (continued)

From the first shift theorem: $\mathcal{L}^{-1}\left\{\dfrac{1}{s-a}\right\} = e^{+at}$

$$\Rightarrow \quad \mathcal{L}^{-1}\left\{\frac{1}{(s+1)(s+2)}\right\} = e^{-1t} - e^{-2t}$$

The second shift theorem states:

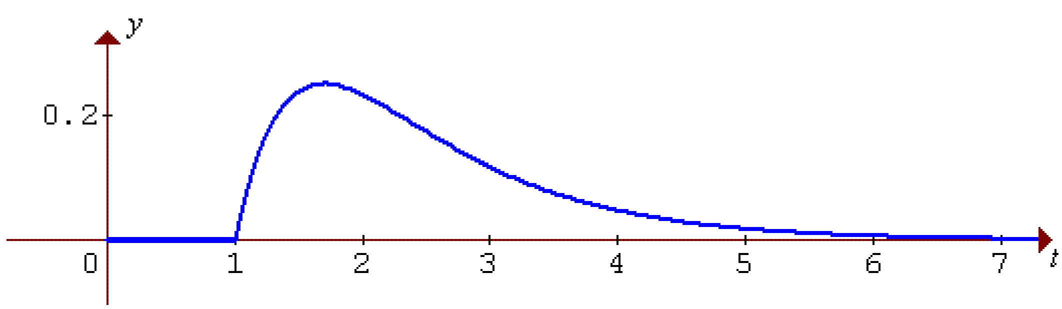
$$\mathcal{L}^{-1}\{F(s)\} = f(t) \quad \Rightarrow \quad \mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)H(t-a)$$

Therefore

$$y(t) = \underline{\underline{\left(e^{-(t-1)} - e^{-2(t-1)}\right)H(t-1)}}$$

or, equivalently,

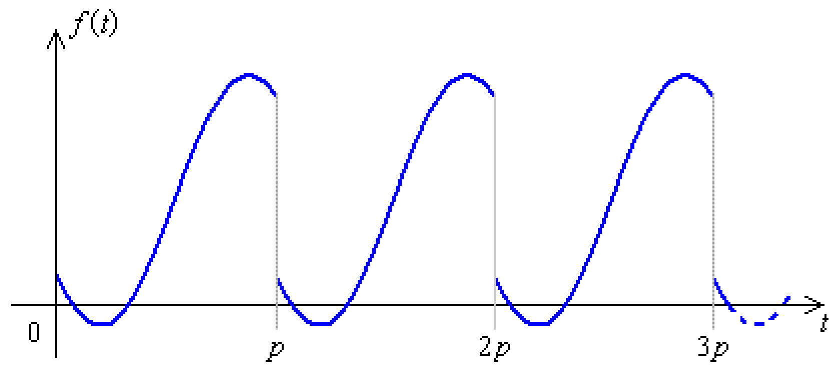
$$y(t) = \begin{cases} 0 & (t<1) \\ e^{-(t-1)} - e^{-2(t-1)} & (t \ge 1) \end{cases}$$



5.09 Periodic Functions

If the constant $p > 0$ and $f(t + p) = f(t)$ for all $t > 0$, then $f(t)$ is a **periodic function** of t, with period p.

Example of a periodic function (with one finite discontinuity in each period):



Define a new set of functions $g_n(t)$, each of which captures only a single period of $f(t)$:

$$g_n(t) = \begin{cases} f(t) & (np < t < (n+1)p) \\ 0 & (\text{otherwise}) \end{cases}$$

Then $\quad f(t) = \sum_{n=0}^{\infty} g_n(t)$

Let $\quad G_n(s) = \mathcal{L}\{g_n(t)\} = \int_0^{\infty} e^{-st} g_n(t)\, dt$

$$= 0 + \int_{np}^{(n+1)p} e^{-st} g_n(t)\, dt + 0 = \int_{np}^{(n+1)p} e^{-st} f(t)\, dt$$

Let $F(s) = \mathcal{L}\{f(t)\}$, then

$$F(s) = \sum_{n=0}^{\infty} \mathcal{L}\{g_n(t)\} = \sum_{n=0}^{\infty} \int_{np}^{(n+1)p} e^{-st} f(t)\, dt$$

Let $\tau = t - np \quad \Rightarrow \quad d\tau = dt$ and

$$F(s) = \sum_{n=0}^{\infty} \int_0^{p} e^{-s(\tau+np)} f(\tau)\, d\tau = \sum_{n=0}^{\infty} e^{-snp} \int_0^{p} e^{-s\tau} f(\tau)\, d\tau$$

$$= \left(\int_0^{p} e^{-s\tau} f(\tau)\, d\tau \right) \underbrace{\left(\sum_{n=0}^{\infty} \left(e^{-sp}\right)^n \right)}_{G.S.,\ a=1,\ r=e^{-sp}} = \frac{1}{1 - e^{-sp}} \times G_0(s)$$

Therefore, for a periodic function $f(t)$ with fundamental period p,

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-sp}} \int_0^p e^{-st} f(t)\, dt$$

<u>Example 5.9.1</u>

Use the formula above to verify the Laplace transform of $f(t) = \sin \omega t$.

$$p = \frac{2\pi}{\omega}$$

$$\Rightarrow \quad F(s) = \frac{1}{1 - e^{-2\pi s/\omega}} \int_0^{2\pi/\omega} e^{-st} \sin \omega t \, dt$$

= ... [after two integrations by parts and algebraic manipulation] ... =

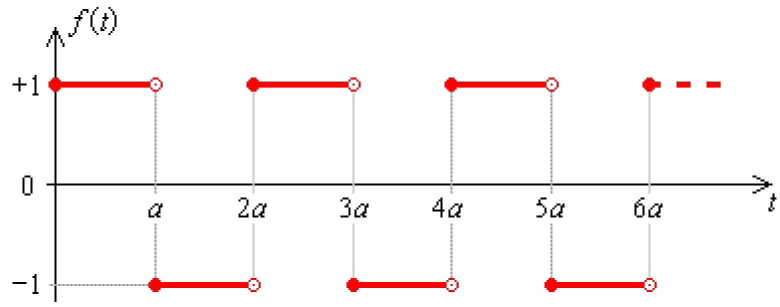
$$= \frac{1}{1 - e^{-2\pi s/\omega}} \left[\frac{-e^{-st}\left(s \sin \omega t + \omega \cos \omega t\right)}{s^2 + \omega^2} \right]_0^{2\pi/\omega}$$

$$= \frac{1}{1 - e^{-2\pi s/\omega}} \left(\frac{-e^{-2\pi s/\omega}\left(0+\omega\right) + 1\left(0+\omega\right)}{s^2 + \omega^2} \right) = \frac{\omega \cancel{\left(1 - e^{-2\pi s/\omega}\right)}}{\cancel{\left(1 - e^{-2\pi s/\omega}\right)} \left(s^2 + \omega^2\right)}$$

Therefore

$$\mathcal{L}\{\sin \omega t\} = \frac{\omega}{s^2 + \omega^2}$$

Example 5.9.2 **The Square Wave**



The square wave is periodic, with fundamental period $p = 2a$.

In the "zeroth" period ($0 < t < 2a$),

$$f(t) = g_0(t) = \begin{cases} +1 & (0 < t < a) \\ -1 & (a < t < 2a) \end{cases}$$

The Laplace transform $F(s)$ of the square wave $f(t)$ then follows.

$$F(s) = \frac{1}{1 - e^{-2as}}\left(\int_0^a e^{-st}(+1)\,dt + \int_a^{2a} e^{-st}(-1)\,dt \right)$$

$$= \frac{1}{1 - e^{-2as}}\left(\left[\frac{e^{-st}}{-s}\right]_0^a + \left[\frac{e^{-st}}{+s}\right]_a^{2a} \right)$$

$$= \frac{1}{1 - e^{-2as}}\left(\frac{1 - e^{-as}}{s} + \frac{e^{-2as} - e^{-as}}{s} \right) = \frac{1}{s}\cdot\left(\frac{1 - 2e^{-as} + e^{-2as}}{1 - e^{-2as}} \right)$$

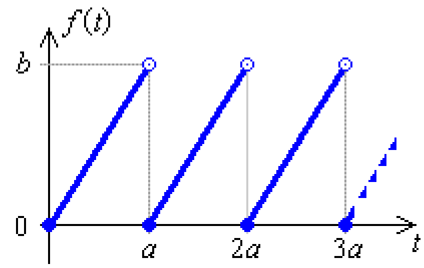
$$= \frac{1}{s}\cdot\frac{\left(1 - e^{-as}\right)^2}{\left(1 - e^{-as}\right)\left(1 + e^{-as}\right)} = \frac{1}{s}\cdot\frac{1 - e^{-as}}{1 + e^{-as}}$$

But $\tanh x \equiv \dfrac{e^x - e^{-x}}{e^x + e^{-x}} \equiv \dfrac{1 - e^{-2x}}{1 + e^{-2x}}$

Set $2x = as$, then

$$\boxed{F(s) = \frac{1}{s}\tanh\left(\frac{as}{2}\right)}$$

Example 5.9.3 **The Saw-tooth Wave**



The saw-tooth wave is periodic, with fundamental period $p = a$.

In the "zeroth" period ($0 < t < a$),

$$f(t) = g_0(t) = \frac{bt}{a}$$

The Laplace transform $F(s)$ of the saw-tooth wave $f(t)$ then follows.

$$F(s) = \frac{1}{1 - e^{-as}} \int_0^a e^{-st} \left(\frac{bt}{a} \right) dt$$

Let $I = \int_0^a e^{-st} t \, dt$

Integration by parts:

D		**I**
t		e^{-st}
	$+$	
1		$-\frac{1}{s} e^{-st}$
	$-$	
0		$+\frac{1}{s^2} e^{-st}$

$$\Rightarrow \quad I = \left[\frac{-e^{-st}}{s^2} (st + 1) \right]_0^a = \frac{-e^{-as}(as + 1) + 1(0 + 1)}{s^2}$$

$$\Rightarrow \quad F(s) = \frac{1}{1 - e^{-as}} \cdot \frac{b}{a} \cdot \frac{1 - (as + 1)e^{-as}}{s^2} = \frac{b}{a} \left(\frac{\cancel{1 - e^{-as}}}{s^2 \left(\cancel{1 - e^{-as}} \right)} + \frac{-as\, e^{-as}}{s^2 \left(1 - e^{-as} \right)} \right)$$

$$\Rightarrow \quad F(s) = \frac{b}{as^2} - \frac{b}{s\left(e^{as} - 1\right)}$$

Example 5.9.4

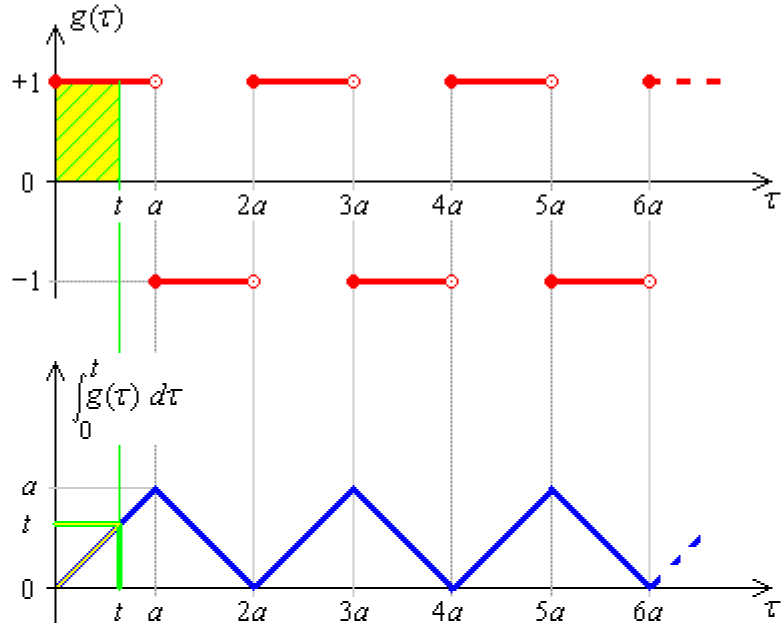
Find the function $f(t)$ whose Laplace transform is $F(s) = \frac{1}{s^2}\tanh\left(\frac{as}{2}\right)$ (where a is a constant).

Let $G(s) = \frac{1}{s}\tanh\left(\frac{as}{2}\right)$,

then $g(t) = \mathcal{L}^{-1}\{ G(s) \}$ = the square wave, of period $p = 2a$.

$$\Rightarrow \quad \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s}G(s)\right\} = \int_0^t g(\tau)\,d\tau$$

This integral can be developed graphically:



Therefore $\underline{\underline{f(t) = \text{triangular wave, period } 2a, \text{ amplitude } a}}$.

Also: $f(t) = t + 2\sum_{n=1}^{\infty}(-1)^n (t-na)H(t-na)$

5.10 The Derivative of a Laplace Transform

Example 5.10.1

Find $F(s) = \mathscr{L}\{\, t \cos \omega t \,\}$.

Method via an ODE:

Let $f(t) = t \cos \omega t$

Then $f'(t) = \cos \omega t - \omega t \sin \omega t$

and $f''(t) = -\omega \sin \omega t - (\omega \sin \omega t + \omega^2 t \cos \omega t) = -2\omega \sin \omega t - \omega^2 f(t)$

Also $f(0) = 0$

$f'(0) = 1 - 0 = 1$

Taking the Laplace transform of this initial value problem,

$$s^2 F(s) - 0 - 1 = -2\omega\left(\frac{\omega}{s^2+\omega^2}\right) - \omega^2 F(s)$$

$$\Rightarrow \left(s^2+\omega^2\right)F(s) = 1 - \frac{2\omega^2}{s^2+\omega^2} = \frac{s^2+\omega^2 - 2\omega^2}{s^2+\omega^2} = \frac{s^2-\omega^2}{s^2+\omega^2}$$

Therefore

$$\boxed{\mathscr{L}\{\, t \cos \omega t \,\} = \frac{s^2-\omega^2}{\left(s^2+\omega^2\right)^2}}$$

$$F(s) = \mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt \quad \Rightarrow \quad F'(s) = \frac{d}{ds}\left(\int_0^\infty e^{-st} f(t)\, dt\right)$$

Using Leibnitz differentiation of an integral [section 1.9]:

$$F'(s) = 0 - 0 + \int_0^\infty \frac{\partial}{\partial s}\left(e^{-st} f(t)\right) dt = \int_0^\infty e^{-st}\left(-t\, f(t)\right) dt$$

Therefore

$$\boxed{\frac{d}{ds}\mathscr{L}\{f(t)\} = -\mathscr{L}\{t\, f(t)\}} \quad \Rightarrow \quad \boxed{\mathscr{L}^{-1}\{F'(s)\} = -t \cdot \mathscr{L}^{-1}\{F(s)\}}$$

Example 5.10.1 (again)

Find $F(s) = \mathscr{L}\{\, t \cos \omega t \,\}$.

Method via the derivative of a transform:

Let $g(t) = \cos \omega t$

$$\Rightarrow \quad G(s) = \mathscr{L}\{g(t)\} = \frac{s}{s^2+\omega^2}$$

$$\Rightarrow \quad \mathscr{L}\{t\,g(t)\} = -\frac{dG}{ds} = -\frac{d}{ds}\left(\frac{s}{s^2+\omega^2}\right)$$

$$= \frac{-\left(1\left(s^2+\omega^2\right) - s\left(2s+0\right)\right)}{\left(s^2+\omega^2\right)^2} = \underline{\underline{\frac{s^2-\omega^2}{\left(s^2+\omega^2\right)^2}}}$$

Example 5.10.2

Find $F(s) = \mathscr{L}\{\, t \sin \omega t \,\}$.

$$\mathscr{L}\{t \sin \omega t\} = -\frac{d}{ds}\left(\mathscr{L}\{\sin \omega t\}\right) = -\frac{d}{ds}\left(\frac{\omega}{s^2+\omega^2}\right)$$

$$= -\omega \frac{d}{ds}\left(\left(s^2+\omega^2\right)^{-1}\right) = -\omega\,(-1)\left(s^2+\omega^2\right)^{-2}(2s)$$

Therefore

$$\boxed{\mathscr{L}\{t \sin \omega t\} = \frac{2\omega s}{\left(s^2+\omega^2\right)^2}}$$

5.11 Convolution

If $\mathcal{L}\{ f(t) \} = F(s)$

and $\mathcal{L}\{ g(t) \} = G(s)$

then the convolution of $f(t)$ and $g(t)$ is denoted by $(f * g)(t)$, is defined by

$$(f*g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau$$

and the Laplace transform of the convolution of two functions is the product of the separate Laplace transforms:

$$\mathcal{L}\{(f*g)(t)\} = F(s)\,G(s)$$

An equivalent identity is

$$\mathcal{L}^{-1}\{F(s)\,G(s)\} = \mathcal{L}^{-1}\{F(s)\}*\mathcal{L}^{-1}\{G(s)\}$$

Convolution is commutative:

$$g*f = \int_0^t f(t-\tau)g(\tau)\,d\tau = \int_0^t f(\tau)g(t-\tau)\,d\tau = f*g$$

Example 5.11.1

Find the inverse Laplace transform of $R(s) = \dfrac{1}{s^2\left(s^2+\omega^2\right)}$.

Let $F(s) = \dfrac{1}{s^2} \quad \Rightarrow \quad f(t) = \mathcal{L}^{-1}\{F(s)\} = t$

and $G(s) = \dfrac{1}{s^2+\omega^2} \quad \Rightarrow \quad g(t) = \mathcal{L}^{-1}\{G(s)\} = \dfrac{\sin\omega t}{\omega}$

$R = F\,G \quad \Rightarrow \quad r = f*g$

$$\Rightarrow \quad r(t) = \int_0^t (t-\tau)\frac{\sin\omega\tau}{\omega}\,d\tau$$

Example 5.11.1 (continued)

Integration by parts:

D		**I**
$t - \tau$		$\dfrac{\sin \omega\tau}{\omega}$
	$+$	
-1		$-\dfrac{\cos \omega\tau}{\omega^2}$
	$-$	
0		$-\dfrac{\sin \omega\tau}{\omega^3}$

$$\Rightarrow \quad r(t) = \left[\frac{-1}{\omega^3} \big((t-\tau)\,\omega \cos \omega\tau \; + \; \sin \omega\tau \big) \right]_{\tau=0}^{\tau=t}$$

$$= \frac{-(0 \; + \; \sin \omega t) \; + \; (\omega t \; + \; 0)}{\omega^3}$$

Therefore

$$\boxed{\mathcal{L}^{-1}\left\{ \frac{1}{s^2 \left(s^2 + \omega^2\right)} \right\} = \frac{\omega t \; - \; \sin \omega t}{\omega^3}}$$

Note:

This inverse transform can also be found using partial fractions or the division of another Laplace transform by s. Both of these alternatives are in example 5.6.2 [on page 5.19].

Example 5.11.2

Find $\mathcal{L}^{-1}\left\{ \dfrac{1}{\left(s^2+\omega^2\right)^2} \right\}$.

The alternative methods that were available in example 5.11.1 are not available here. The only obvious way to proceed is via convolution.

Let $R(s) = \dfrac{1}{\left(s^2+\omega^2\right)^2}$

Let $F(s) = \dfrac{1}{s^2+\omega^2} = G(s) \quad \Rightarrow \quad f(t) = g(t) = \dfrac{\sin\omega t}{\omega}$

$R = F\,G \quad \Rightarrow \quad r = f * g$

$$\Rightarrow \quad r(t) = \int_0^t \frac{\sin\omega(t-\tau)}{\omega} \cdot \frac{\sin\omega\tau}{\omega}\,d\tau$$

But $\sin A \sin B = ½ (\cos(A-B) - \cos(A+B))$

$$\Rightarrow \quad \sin\omega(t-\tau)\sin\omega\tau = \tfrac{1}{2}\left(\cos\omega(t-2\tau) - \cos\omega t\right)$$

$$\Rightarrow \quad r(t) = \frac{1}{2\omega^2}\int_0^t \left(\cos\omega(t-2\tau) - \cos\omega t\right)d\tau$$

$$= \frac{1}{2\omega^2}\left[\frac{\sin\omega(t-2\tau)}{-2\omega} - \tau\cos\omega t\right]_{\tau=0}^{\tau=t}$$

$$= \frac{1}{2\omega^2}\left(\frac{-\sin\omega t}{-2\omega} - t\cos\omega t - \frac{\sin\omega t}{-2\omega} + 0\right)$$

Therefore

$$\boxed{\mathcal{L}^{-1}\left\{ \frac{1}{\left(s^2+\omega^2\right)^2} \right\} = \frac{\sin\omega t - \omega t\cos\omega t}{2\omega^3}}$$

Transfer Function

For a second order linear ordinary differential equation with constant coefficients, when both initial conditions are zero, the complete solution can be expressed as a convolution.

$$y'' + by' + cy = r(t) \quad \text{and} \quad y(0) = y'(0) = 0$$

$$\Rightarrow \quad (s^2 + bs + c)\, Y(s) = R(s)$$

$$\Rightarrow \quad Y(s) = Q(s)\, R(s)\,, \text{ where } Q(s) \text{ is the } \textbf{transfer function}:$$

$$Q(s) = \frac{1}{s^2 + bs + c}$$

The complete solution is just $y(t) = q(t) * r(t)$, where $q(t) = \mathscr{L}^{-1}\left\{ \dfrac{1}{s^2 + bs + c} \right\}$.

<u>Example 5.11.3</u>

A unit impulse is delivered at time $t = 1$ to an harmonic system (with $c/m = 4$) that is in equilibrium until that instant. Find the response $y(t)$.

$$y'' + 4y = \underbrace{\delta(t-1)}_{r(t)} \quad ; \qquad y(0) = y'(0) = 0$$

$$\Rightarrow \quad (s^2 + 4)\, Y(s) = R(s)$$

$$\Rightarrow \quad Y(s) = Q(s)\, R(s)\,, \text{ where } Q(s) = \frac{1}{s^2 + 4}$$

$$\Rightarrow \quad q(t) = \tfrac{1}{2} \sin 2t$$

$$\Rightarrow \quad y(t) = q * r = \tfrac{1}{2} \sin 2t * \delta(t-1) = \frac{1}{2} \int_0^t \sin 2(t-\tau)\, \delta(\tau - 1)\, d\tau$$

The range of integration includes the infinite Dirac spike at $\tau = 1$ only if $t \ge 1$.
Using the sifting property of the Dirac delta function,

$$y(t) = \left\{ \begin{matrix} 0 & (t < 1) \\ \tfrac{1}{2} \sin 2(t-1) & (t \ge 1) \end{matrix} \right\} = \underline{\underline{\tfrac{1}{2} \sin 2(t-1)\, H(t-1)}}$$

Some More Identities involving Convolution

$$1 * y = \int_0^t 1 \times y(\tau)\,d\tau = \int_0^t y(\tau)\,d\tau$$

$$\Rightarrow \quad \mathcal{L}\left\{ \int_0^t y(\tau)\,d\tau \right\} = \mathcal{L}\{1 * y\} = \mathcal{L}\{1\} \cdot \mathcal{L}\{y\} = \frac{1}{s}\mathcal{L}\{y(t)\}$$

[This is an identity that we saw established in another way in section 5.06.]

$$(\delta * f)(t) = \int_0^t f(\tau)\,\delta(t-\tau)\,d\tau = f(t) \qquad [\text{sifting property}]$$

Therefore

$$\boxed{\delta * f = f}$$

and

$$\delta(t-a) * f(t) = \int_0^t f(t-\tau)\,\delta(\tau-a)\,d\tau = \begin{cases} f(t-a) & (0 < a < t) \\ 0 & (a > t) \end{cases}$$

Therefore

$$\boxed{\delta(t-a) * f(t) = f(t-a)\,H(t-a)}$$

This leads to an alternate proof of the second shift theorem:

$$\mathcal{L}\{f(t-a)H(t-a)\} = \mathcal{L}\{\delta(t-a) * f(t)\}$$
$$= \mathcal{L}\{\delta(t-a)\} \cdot \mathcal{L}\{f(t)\} = \mathcal{L}\{f(t)\}e^{-as}$$

Integro-Differential Equations

Example 5.11.4

Find the solution $y(t)$ of the integro-differential equation

$$y(t) = t^2 + 1 - 9\int_0^t (t-x)\,y(x)\,dx$$

$y(t) = t^2 + 1 - 9\,t * y(t)$

Let $Y(s) = \mathscr{L}\{\,y(t)\,\}$, then

$$Y(s) = \frac{2}{s^3} + \frac{1}{s} - 9\cdot\frac{1}{s^2}\cdot Y(s)$$

$$\Rightarrow \left(\frac{s^2+9}{s^2}\right)Y(s) = \frac{2+s^2}{s^3} \qquad \Rightarrow \quad Y(s) = \frac{2+s^2}{s^{\not 3}}\cdot\frac{\not s^2}{s^2+9}$$

$$\frac{s^2+2}{s\left(s^2+9\right)} = \frac{a}{s} + \frac{bs + c(3)}{s^2+3^2}$$

Cover-up rule:

$$a = \frac{0^2+2}{\boxed{\times}\left(0^2+9\right)} = \frac{2}{9}$$

$$s^2 + 2 = \tfrac{2}{9}\left(s^2+9\right) + \left(bs + 3c\right)s$$

Matching coefficients:

s^2: $\quad 1 = 2/9 + b \quad \Rightarrow \quad b = 7/9$

s^1: $\quad 0 = 0 + 3c \quad \Rightarrow \quad c = 0$

$$\Rightarrow \quad Y(s) = \frac{1}{9}\left(\frac{2}{s} + \frac{7s}{s^2+3^2}\right) \qquad \Rightarrow \quad y(t) = \underline{\underline{\frac{1}{9}\left(2 + 7\cos 3t\right)}}$$

Example 5.11.5

Solve the system of simultaneous integral equations

$$\left\{ \begin{array}{c} 4i_1 + 12\int_0^t (i_1 - i_2)\,d\tau = 1 \\ i_2 + 2\int_0^t i_2\,d\tau = 2\int_0^t (i_1 - i_2)\,d\tau \end{array} \right\}$$

Let $I_1(s) = \mathscr{L}\{i_1(t)\}, \quad I_2(s) = \mathscr{L}\{i_2(t)\}$.

Taking the Laplace transform of the entire system of integral equations,

$$\left\{ \begin{array}{c} 4I_1 + 12\frac{(I_1 - I_2)}{s} = \frac{1}{s} \\ I_2 + 2\frac{I_2}{s} = 2\frac{(I_1 - I_2)}{s} \end{array} \right\}$$

$$\left\{ \begin{array}{c} 4sI_1 + 12(I_1 - I_2) = 1 \\ sI_2 + 2I_2 - 2(I_1 - I_2) = 0 \end{array} \right\}$$

$$\Rightarrow \underbrace{\begin{bmatrix} 4s+12 & -12 \\ -2 & s+4 \end{bmatrix}}_{A} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\det A = (4s+12)(s+4) - (-2)(-12) = \ldots = 4(s^2 + 7s + 6)$$

$$\Rightarrow \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \frac{1}{4(s+1)(s+6)} \begin{bmatrix} s+4 & 12 \\ 2 & 4s+12 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \frac{1}{4(s+1)(s+6)} \begin{bmatrix} s+4 \\ 2 \end{bmatrix}$$

Example 5.11.5 (continued)

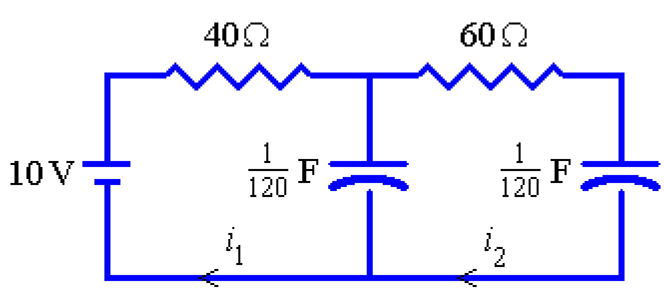
$$\Rightarrow \quad I_1 = \frac{s+4}{4(s+1)(s+6)} = \frac{1}{4}\left(\frac{\left(\frac{-1+4}{-1+6}\right)}{s+1} + \frac{\left(\frac{-6+4}{-6+1}\right)}{s+6}\right)$$

$$\Rightarrow \quad i_1(t) = \underline{\underline{\tfrac{1}{20}\left(3e^{-t} + 2e^{-6t}\right)}}$$

$$\text{and} \quad I_2 = \frac{2}{4(s+1)(s+6)} = \frac{1}{2}\left(\frac{\left(\frac{1}{-1+6}\right)}{s+1} + \frac{\left(\frac{1}{-6+1}\right)}{s+6}\right)$$

$$\Rightarrow \quad i_2(t) = \underline{\underline{\tfrac{1}{10}\left(e^{-t} - e^{-6t}\right)}}$$

These are the currents in the circuit



with initial conditions $i_1(0) = 1/4, \; i_2(0) = 0$.

END OF CHAPTER 5