1) The joint probability mass function $p(x, y)$ for random quantities X, Y is defined by the table:

		Y			
	$p(x, y)$	–1	0	1	
	–1	.06	.09	.15	
X	0	.10	.15	.25	
	1	.04	.06	.10	

(a) Verify that $p(x, y)$ is a valid probability mass function. [2]

(b) Find the correlation coefficient $\rho_{X,Y}$. [7]

(c) Are the random quantities X, Y independent? Why or why not? [4]

(a) Completing the marginal p.m.f.s:

		Y			
	$p(x, y)$	–1	0	1	$p_X(x)$
	–1	.06	.09	.15	**.30**
X	0	.10	.15	.25	**.50**
	1	.04	.06	.10	**.20**
	$p_Y(y)$	**.20**	**.30**	**.50**	**1**

- All nine values of $p(x, y)$ are positive.
- Clearly $\sum_{x=-1}^{1}\sum_{y=-1}^{1} p(x, y) = 1$ (coherence).

Therefore $p(x, y)$ is a valid probability mass function.

(b) $\mathrm{E}[X] = \sum_{x=-1}^{1} x \cdot p_X(x) = -1\times.30 + 0\times.50 + 1\times.20 = -0.10$

$\mathrm{E}[Y] = \sum_{y=-1}^{1} y \cdot p_Y(y) = -1\times.20 + 0\times.30 + 1\times.50 = 0.30$

To the continuation of this question

1 (b) (continued)

$$\mathrm{E}[XY] = \sum_{x=-1}^{1}\sum_{y=-1}^{1} xy \cdot p(x,y) =$$

$$= -1\times-1\times.06 + 0 + -1\times1\times.15 + 0+0+0 + 1\times-1\times.04 + 0 + 1\times1\times.10 = -0.03$$

$$\mathrm{Cov}(X,Y) = \mathrm{E}[XY] - \mathrm{E}[X]\cdot\mathrm{E}[Y] = -0.03 - 0.30\times(-0.10) = -0.03+0.03 = 0$$

$$\mathrm{Cov}(X,Y) = 0 \quad \Rightarrow$$

$$\boxed{\rho = 0}$$

Note that there is no need to evaluate V[X] or V[Y].
[The values are V[X] = 0.49 and V[Y] = 0.61.]
An Excel spreadsheet is available to illustrate this solution.

(c) $\rho = 0 \not\Rightarrow$ independence!

We must therefore check that $p(x,y) = p_X(x)\cdot p_Y(y) \quad \forall (x,y)$.

$p_X(-1)\cdot p_Y(-1) = .3\times.2 = .06 = p(-1,-1)$ ✓

$p_X(-1)\cdot p_Y(0) = .3\times.3 = .09 = p(-1,0)$ ✓

$p_X(-1)\cdot p_Y(1) = .3\times.5 = .15 = p(-1,1)$ ✓

$p_X(0)\cdot p_Y(-1) = .5\times.2 = .10 = p(0,-1)$ ✓

$p_X(0)\cdot p_Y(0) = .5\times.3 = .15 = p(0,0)$ ✓

$p_X(0)\cdot p_Y(1) = .5\times.5 = .25 = p(0,1)$ ✓

$p_X(1)\cdot p_Y(-1) = .2\times.2 = .04 = p(1,-1)$ ✓

$p_X(1)\cdot p_Y(0) = .2\times.3 = .06 = p(1,0)$ ✓

$p_X(1)\cdot p_Y(1) = .2\times.5 = .10 = p(1,1)$ ✓

Therefore

YES

2) Lamps from a certain factory are known to have lifetimes T that are independent random quantities following an exponential distribution with a mean lifetime of 10,000 hours.

(a) Show that the probability p that a randomly chosen lamp has a lifetime exceeding 23,026 hours is 0.100 00, correct to five decimal places. [4]

(b) A random sample of ten such lamps is tested. Let X be the number of lamps in this sample that have lifetimes exceeding 23,026 hours. Does X follow a binomial distribution exactly, approximately or not at all? Justify your answer. [2]

(c) Assume that $p = 0.1$ exactly. Write down the value of $\text{E}[X]$. [2]

(d) Find $\text{P}[X < 2]$. [3]

(e) Another random sample of 100 lamps is tested. Estimate the probability that the sample mean lifetime $\overline{T}$ will be less than 9,000 hours. [4]

(a) $\mu = 10000 \quad \Rightarrow \quad \lambda = \dfrac{1}{10000}$

$$p = \text{P}[T > 23026] = e^{-\lambda t} = \exp\left(-\frac{23026}{10000}\right) = e^{-2.3026} = 0.099998\ldots$$

Therefore $p = 0.100\,00$, correct to 5 d.p.

(b) Let “success” = “lamp has lifetime exceeding 23,026 hours”

- Each trial (lamp) has a complementary pair of outcomes (success or not);
- P[success] = constant = p ;
- The trials are independent (each lamp has the same P[success], independently of the others);
- The sample size is fixed (n = 10).

Therefore the p.m.f. of X is **binomial exactly**.

(c) $\text{E}[X] = n\,p = 10 \times .1 = \boxed{1}$

(d) $\text{P}[X < 2] = \text{P}[X \le 1] = \text{P}[X = 0] + \text{P}[X = 1] = b(0; 10, .1) + b(1; 10, .1)$

$$= (.9)^{10} + {}^{10}C_1 (.1)^1 (.9)^9 = .348\,678\,4401 + .387\,420\,4890 = .736\,098\ldots$$

Therefore, correct to 3 s.f.,

$$\boxed{\text{P}[X < 2] = \mathbf{.736}}$$

To the continuation of this question ☞

2 (e) $n = 100$. By the central limit theorem, $\overline{T}$ will follow a normal distribution to an excellent approximation.

$$\mathrm{E}[T] = \mu = 10000 \quad \Rightarrow \quad \sqrt{\mathrm{V}[T]} = \sigma = 10000$$

$$\Rightarrow \quad \mathrm{E}\left[\overline{T}\right] = 10000 \quad \text{and} \quad \sqrt{\mathrm{V}\left[\overline{T}\right]} = \frac{\sigma}{\sqrt{n}} = \frac{10000}{10} = 1000$$

$$\Rightarrow \quad \overline{T} \sim \mathrm{N}\left(10\,000, (1000)^2\right) \quad \text{to an excellent approximation.}$$

$$\Rightarrow \quad \mathrm{P}\left[\overline{T} < 9\,000\right] = \mathrm{P}\left[Z < \frac{9\,000 - 10\,000}{1000}\right] = \Phi(-1.00) = .158\,66\ldots \quad \Rightarrow$$

$$\boxed{\mathrm{P}\left[\overline{T} < 9\,000\right] \approx .159}$$

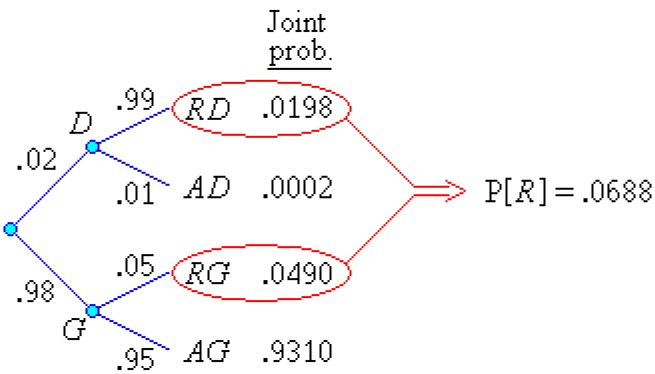
3) Two percent of all items from a production line are known to be defective. [10]
A quality control process rejects a defective item 99% of the time and it rejects a good (non-defective) item 5% of the time.

Given that the quality control process has just rejected an item, find the odds that the item is, indeed, defective.

Let D = "item is defective", $G = \overline{D}$ = "item is good (not defective)",
R = "item is rejected" and $A = \overline{R}$ = "item is accepted", then

From the question, $\text{P}[D] = .02$, $\text{P}[R \mid D] = .99$ and $\text{P}\left[R \mid \overline{D}\right] = \text{P}[R \mid G] = .05$

We are required to find the odds associated with $\text{P}[D \mid R]$.



$$\Rightarrow \quad \text{P}[D \mid R] = \frac{\text{P}[D \cap R]}{\text{P}[R]} = \frac{.0198}{.0688} = \frac{99}{344}$$

OR

using Bayes' theorem directly,

$$\text{P}[D \mid R] = \frac{\text{P}[R \mid D] \cdot \text{P}[D]}{\text{P}[R \mid D] \cdot \text{P}[D] + \text{P}[R \mid G] \cdot \text{P}[G]}$$

$$= \frac{.99 \times .02}{.99 \times .02 + .05 \times .98} = \frac{.0198}{.0198 + .0490} = \frac{.0198}{.0688} = \frac{99}{344}$$

$$r = \frac{p}{1-p} = \frac{99}{344} \times \frac{344}{245} = 99 : 245 \text{ on , or}$$

$r = 245 : 99$ against

An Excel spreadsheet is available to illustrate this solution.

4) A cumulative distribution function $F(x)$ of a continuous variable x is defined by

$$F(x) = \begin{cases} 0 & (x<0) \\ 21x^5 - 35x^6 + 15x^7 & (0 \le x \le 1) \\ 1 & (x>1) \end{cases}$$

(a) Evaluate $\text{P}\left[X > \frac{1}{2}\right]$ exactly. Leave your answer as a fraction. [6]

(b) Find the probability density function (p.d.f.) for this c.d.f. in its simplest form; [that is, factor $f(x)$ as much as possible.] [6]

BONUS QUESTION:

(c) Find the population mean μ as a fraction reduced to its lowest terms. [+3]

(a) $\text{P}\left[X > \frac{1}{2}\right] = 1 - \text{P}\left[X \le \frac{1}{2}\right] = 1 - F\left(\frac{1}{2}\right) = 1 - \left(21\left(\frac{1}{2}\right)^5 - 35\left(\frac{1}{2}\right)^6 + 15\left(\frac{1}{2}\right)^7\right)$

$= 1 - \left(\frac{1}{2}\right)^7 (21\times4 - 35\times2 + 15) = \dfrac{128 - (84 - 70 + 15)}{128} = \dfrac{128 - 29}{128} \Rightarrow$

$$\text{P}\left[X > \tfrac{1}{2}\right] = \mathbf{\frac{99}{128}}$$

(b) $f(x) = \dfrac{dF}{dx} \quad \forall x$ Clearly $f(x) = 0$ for $x < 0$ and for $x > 1$.

For $0 < x < 1$, $\dfrac{dF}{dx} = 105x^4 - 210x^5 + 105x^6 = 105x^4\left(1 - 2x + x^2\right)$

$f(x)$ is continuous both at $x = 0$ and at $x = 1$. Therefore the p.d.f. is

$$f(x) = 105x^4(1-x)^2, \quad (0 \le x \le 1)$$

(c) $\mu = \displaystyle\int_{-\infty}^{\infty} x \cdot f(x)\,dx = 0 + \int_0^1 \left(105x^5 - 210x^6 + 105x^7\right)dx + 0$

$= \left[\dfrac{35x^6}{2} - 30x^7 + \dfrac{105x^8}{8}\right]_0^1 = \dfrac{140 - 240 + 105}{8} - 0 \Rightarrow$

$$\mu = \frac{5}{8}$$

[This probability distribution is actually Beta($\alpha = 5$, $\beta = 3$, $A = 0$, $B = 1$).]

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