ENGI 3425 Mathematics for Civil Engineering I

Problem Set 6 Questions

(Sections 7.1 - 7.4 – Partial Derivatives, Differentials, Jacobian)

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1. Find $\dfrac{\partial^2 u}{\partial x^2}$ and $\dfrac{\partial^2 u}{\partial z\,\partial t}$ for the function $u(x, y, z, t)$ defined by

$$u^2 = x^2 + y^2 + z^2 - t^2$$

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2. Given $z = \sin(x - ct)$, find $\dfrac{\partial^2 z}{\partial x^2}$ and $\dfrac{\partial^2 z}{\partial t^2}$.

Hence show that z satisfies the partial differential equation (P.D.E.)

$$\frac{\partial^2 z}{\partial t^2} = c^2 \frac{\partial^2 z}{\partial x^2}$$

This P.D.E. is called the wave equation.

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3. A pyramid with a square base of side b and a vertical height h has a total exposed surface area of

$$S = b\sqrt{4h^2 + b^2}$$

and an enclosed volume of

$$V = \frac{1}{3} b^2 h$$

(a) Find the rate and manner (increasing or decreasing) in which S and V are changing at the instant when b = 15 m, h = 10 m, h is increasing at a rate of 2 m s^{-1} and b is decreasing at a rate of 1 m s^{-1}.

(b) Use differentials to estimate the percentage change $(\Delta V / V) \times 100\%$ in the enclosed volume when h increases by 3% and b decreases by 2%.
[Hint: Express dV in terms of db and dh, then divide this equation by V in order to express the relative change $\Delta V/V$ ($\approx dV/V$) in terms of the relative changes db/b and dh/h.]

(c) Show that the exact relative change in the volume of the pyramid, when the base b decreases by 2% and the height h increases by 3%, is a decrease of 1.0788%.

[Hint: Evaluate $100\% \times \dfrac{V(b + \Delta b,\ h + \Delta h) - V(b, h)}{V(b, h)}$.]

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4. The displacement of a uniform beam of length L in a vertical plane is represented by the dependent variable u. For any distance x from one end of the beam and at any time t, the displacement function is

$$u(x,t) = (3\cos\beta x + 5\cosh\beta x)\sin\beta^2 ct$$

(where β and c are constants).

(a) Verify that this function satisfies the fourth order partial differential equation

$$\frac{\partial^2 u}{\partial t^2} + c^2 \frac{\partial^4 u}{\partial x^4} = 0$$

(b) In part (a), what must the dimensions (kg - m - s) of the constant c be in order for the P.D.E. to be dimensionally consistent?

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5. Find the Jacobian of the transformation from the (x, y) to the (r, s) system, where

$$x^2 + y^2 + s^2 + r^4 = 1 \quad \text{and} \quad x + 2y - 4r + 3s^2 = 7$$

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6. Find the Jacobian of the transformation from the (x, y, z) to the (r, s, t) system, where

$$x = rs\cos t, \quad y = rs\sin t \quad \text{and} \quad z = \frac{1}{2}\left(r^2 - s^2\right)$$

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7. Find $\dfrac{\partial x}{\partial w}$ when $x = x(z, w)$ and $y = y(z, w)$ are defined implicitly by

$$x^3 + y + z^2 + w^{-2} = 1 \qquad x^2 + 2y - 4z^{-1} + 3w^2 = 7$$

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8. The function $s(t)$ is the distance between two moving particles A at $\left(x_1(t), y_1(t)\right)$ and B at $\left(x_2(t), y_2(t)\right)$ in $\mathbb{R}^2$.

(a) Use the chain rule to deduce that the rate at which the two points are separating from each other is

$$\frac{ds}{dt} = \frac{1}{s}\left((x_2 - x_1)\left(\frac{dx_2}{dt} - \frac{dx_1}{dt}\right) + (y_2 - y_1)\left(\frac{dy_2}{dt} - \frac{dy_1}{dt}\right)\right)$$

(b) Particle A is moving east, parallel to the x-axis with constant speed 2 m s^{-1}.
Particle B is moving north-east, parallel to the line $y = x$ with constant speed $\sqrt{2}$ ms^{-1}.
Find the rate at which the two particles are separating when A is at $(1, 3)$ and B is at $(4, -1)$.

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9. The lengths of the sides of a square are quoted to be (5.2 ± 0.1) cm.
The height of a prism with this square cross section is quoted to be (10.1 ± 0.1) cm.
Use differentials to estimate the maximum relative error in the calculation of the volume V of the prism. Hence estimate the maximum absolute error in V.

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10. Find $\frac{\partial u}{\partial y}$ (in terms of x, y and z only), where

$$u = e^{\left(s^3\right)} + \ln\left(r s^2\right), \qquad r = x^2 + y^3 + z^4 , \quad s = x^2 \cos z$$

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