3. Conic Sections

Contents:

3.1 Standard Form

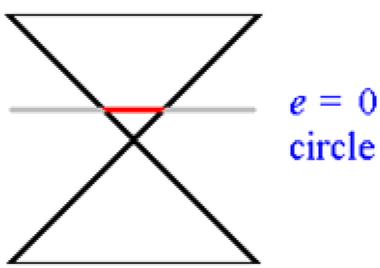
All members of the family of curves known as conic sections can be generated, (as the name implies), from the intersections of a plane and a double cone. The Cartesian equation of any conic section is a second order polynomial in x and y. The only cases that we shall consider in this section are such that any axis of symmetry is parallel to a coordinate axis. For all such cases, the Cartesian equation is of the form

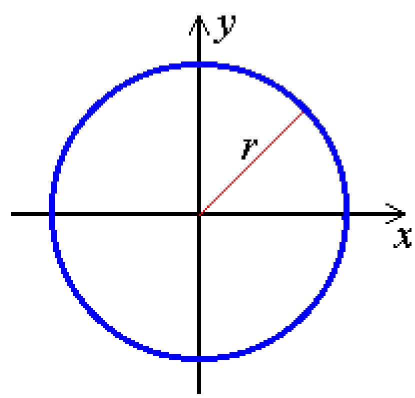
$$Ax^2 + Cy^2 + Dx + Ey + F = 0$$

where A, C, D, E and F are constants. There is no "xy" term, so $B = 0$.

The slope of the intersecting plane is related to the **eccentricity**, e of the conic section.

Circle





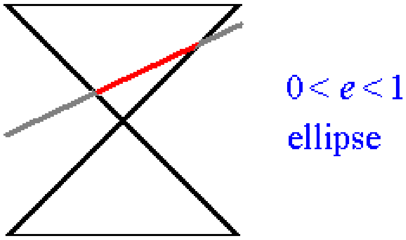
$$x^2 + y^2 = r^2$$

or, if the centre is at (h, k), $(x-h)^2 + (y-k)^2 = r^2$

A parametric form for a circle, centre at the origin, radius r is

$$(x, y) = (r\cos\theta,\ r\sin\theta), \qquad (0 \le \theta < 2\pi).$$

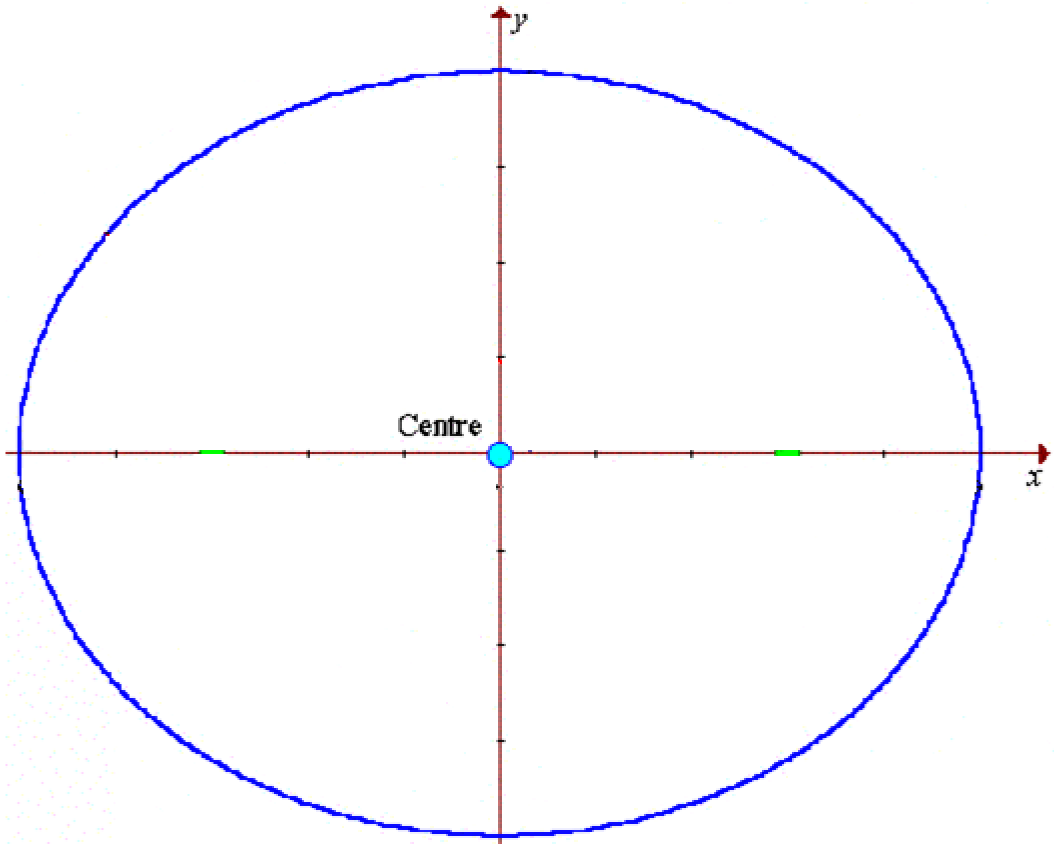
Ellipse



Cartesian equation:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where $b^2 = a^2\left(1 - e^2\right)$



The circle is clearly a special case of the ellipse, with $e = 0$ and $b = a = r$.

The longest diameter is the major axis ($2a$). The shortest diameter is the minor axis ($2b$).

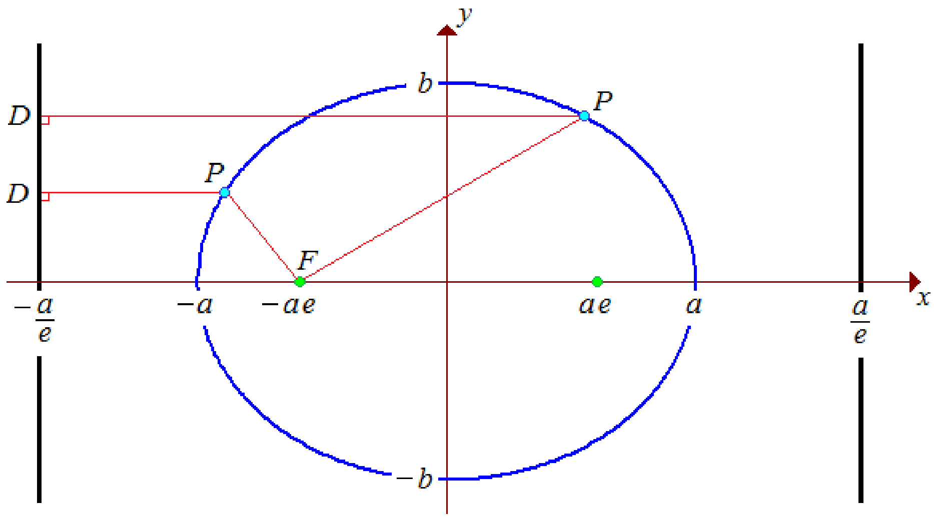
If a mirror is made in the shape of an ellipse, then all rays emerging from one focus will, after reflection, converge on the other focus.

A parameterization for the ellipse is $\vec{\mathbf{r}}(\theta) = a\cos\theta\,\hat{\mathbf{i}} + b\sin\theta\,\hat{\mathbf{j}}, \quad (0 \le \theta < 2\pi)$.

Ellipse (continued)

The ellipse also possesses directrices at $x = \pm \frac{a}{e}$, parallel to the minor axis.

Let *P* be any point on the ellipse, let *F* be a focus and let *D* be point on the directrix nearer to *F* such that the line segment *PD* is parallel to the major axis.



Then the eccentricity *e* is defined as $e = \frac{PF}{PD}$

The ellipse shown here has an eccentricity $e = 0.6$.

The directrices of a circle are at infinity.

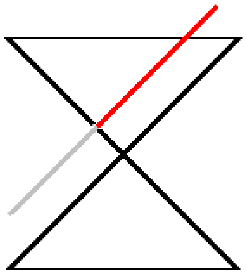
Another feature of the ellipse is that the sum of the distances to the foci is constant for all points *P* on the ellipse.

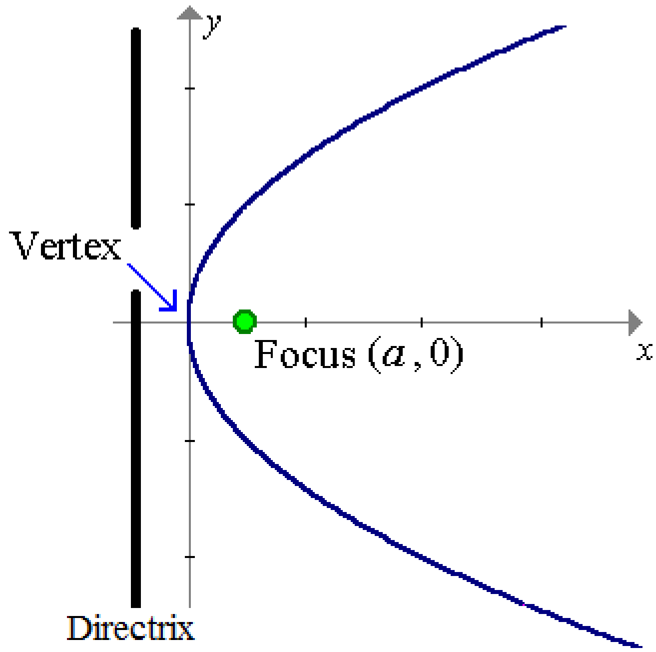
$$PF_1 + PF_2 = 2a$$

If the ellipse is translated so that the centre moves from $(0,0)$ to (h,k) then the equation changes to

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

Parabola



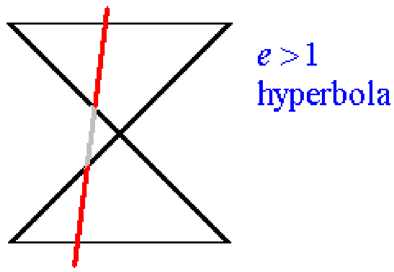


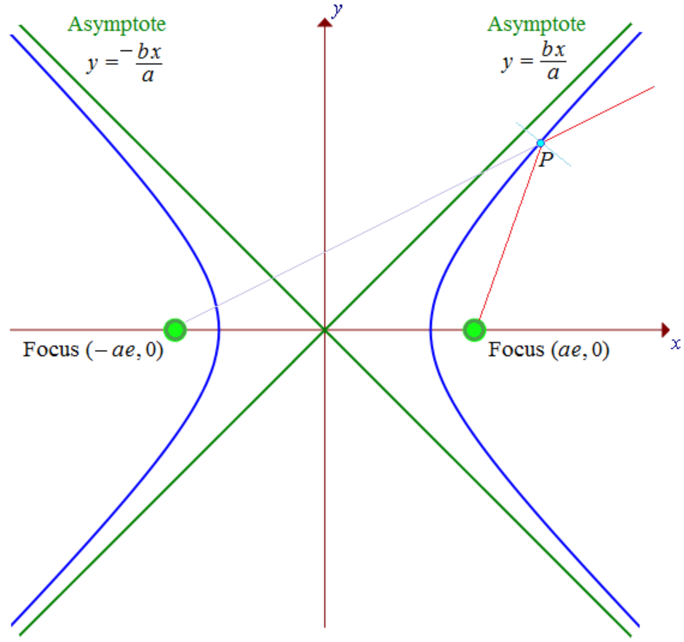
$y^2 = 4ax$

One vertex is at the origin, one directrix is at $x = -a$
and one focus is at $(a, 0)$.
The centre and the other vertex, focus and directrix are at infinity.

If a mirror is made in the shape of a parabola, then all rays emerging from the focus will, after reflection, travel in parallel straight lines to infinity (where the other focus is). The primary mirrors of most telescopes follow a paraboloid shape.

Hyperbola





$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

where $b^2 = a^2\left(e^2 - 1\right)$

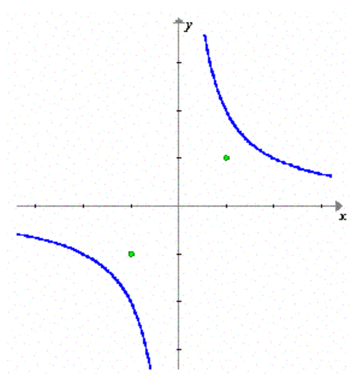
The hyperbola has two separate branches.
As the curve retreats towards infinity, the curve approaches the asymptotes

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0, \quad \left(\Rightarrow \quad y = \pm\frac{bx}{a}\right).$$

The distance between the two vertices is the major axis ($2a$).
If a mirror is made in the shape of an hyperbola, then all rays emerging from one focus will, after reflection, appear to be diverging from the other focus.

Circles and ellipses are closed curves. Parabolas and hyperbolas are open curves.

A special case of the hyperbola occurs when the eccentricity is $e = \sqrt{2}$ and it is rotated 45° from the standard orientation. The asymptotes line up with the coordinate axes, the graph lies entirely in the first and third quadrants and the Cartesian equation is $xy = k$.



This is the **rectangular hyperbola**.

Degenerate conic sections arise when the intersecting plane passes through the apex of the cone. Two cases are:

$0 \le e < 1$: $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 0$$

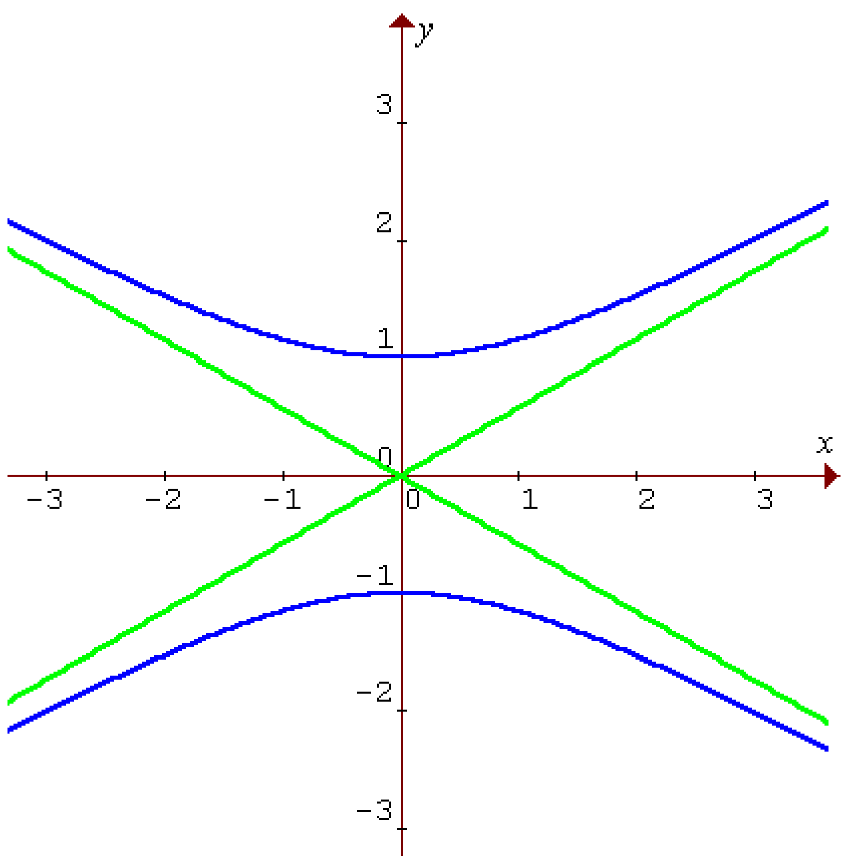
$e > 1$: $$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$$

Another degenerate case is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = -1$$

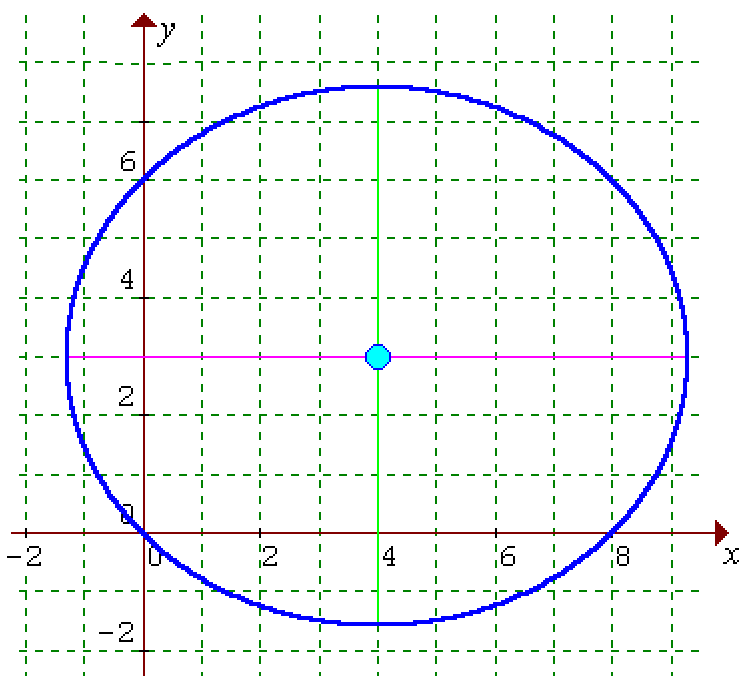
Example 3.1.1

Classify the conic section whose Cartesian equation is $3y^2 = x^2 + 3$.



Example 3.1.2

Classify the conic section whose Cartesian equation is $21x^2 + 28y^2 = 168x + 168y$.



<u>Example 3.1.3</u>

Classify the conic section whose Cartesian equation is $x^2 + y^2 + 2x - 4y + 6 = 0$.

Generally, for non-degenerate conic sections with $B = 0$, the major axis is parallel to a coordinate axis and

$A = C \quad \Rightarrow \quad$ **circle**,

otherwise

A, *C* same sign $\Rightarrow$ **ellipse**

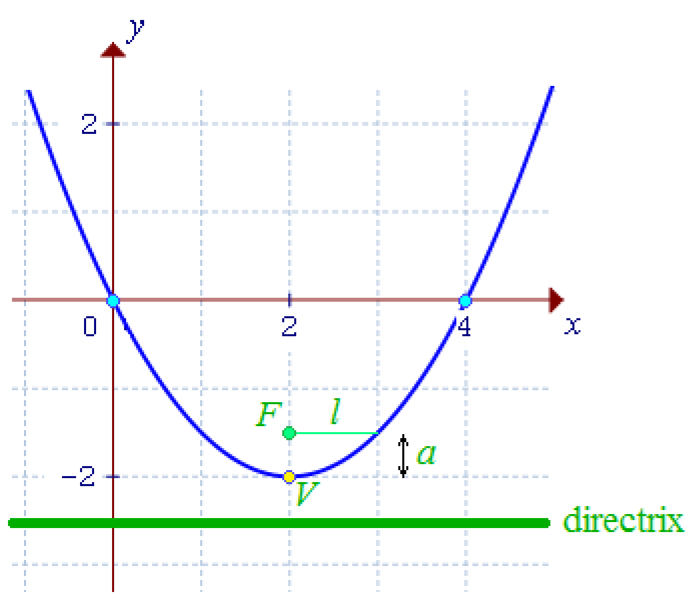
one of *A*, *C* zero $\Rightarrow$ **parabola**

A, *C* opposite sign $\Rightarrow$ **hyperbola**

Example 3.1.4

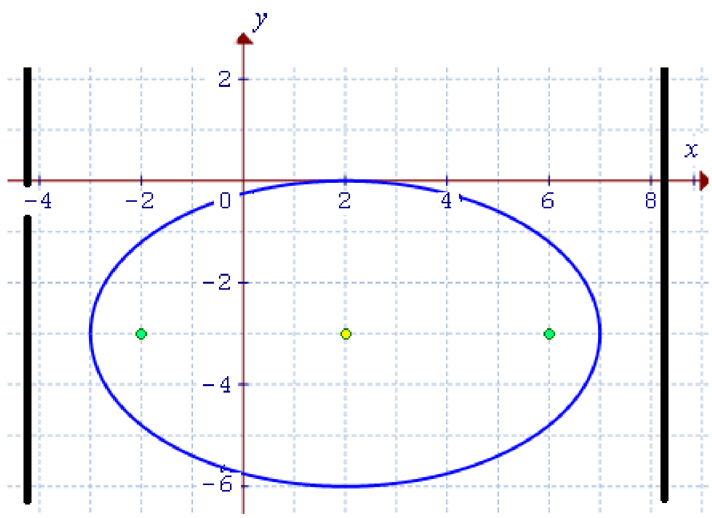
Find the equation of the vertically-oriented parabola that passes through the points $(0,0)$, $(2,-2)$ and $(4,0)$ **and** find the location of the vertex and focus **and** the equation of the directrix. Identify the vertex, focus, semi latus rectum and directrix on a sketch of the parabola.

Example 3.1.4 (continued)



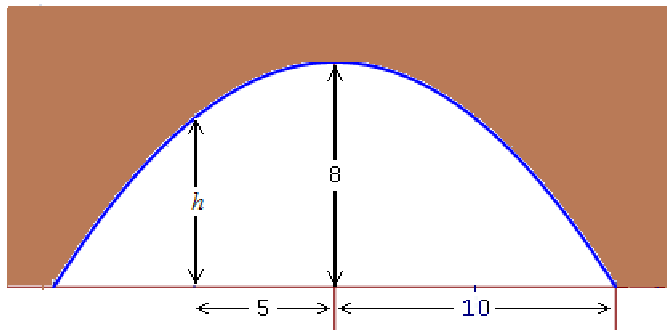
Example 3.1.5

Find the Cartesian equation of the ellipse with foci at $(-2,-3)$ and $(6,-3)$ and eccentricity 0.8. Also find the equations of the directrices.



Example 3.1.6

A bridge is supported by a parabolic arch which has a maximum height of 8 m and is 20 m wide at ground level. A road passes directly under the vertex of the arch. What is the height of the arch 5 m away from the centreline of the road?



3.2 General Conic Sections

If the major axis of an ellipse, parabola or hyperbola is not parallel to one of the coordinate axes, then the Cartesian equation of the conic section will include a term in *xy*. The general equation of a conic section is

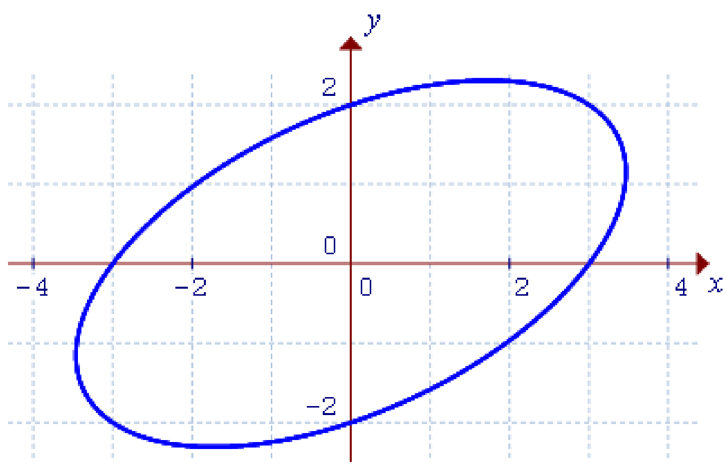
$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

The conic section can be classified as follows:
Find the discriminant $\Delta = B^2 - 4AC$
If the conic section is not degenerate, then
$\Delta < 0 \quad \Rightarrow$ ellipse (circle if also $B = 0$ and $C = A$)
$\Delta = 0 \quad \Rightarrow$ parabola
$\Delta > 0 \quad \Rightarrow$ hyperbola (rectangular if also $C = -A$)
Note that opposite signs for *A*, *C* guarantee that a non-degenerate conic is a hyperbola.

Example 3.2.1

Classify the conic section whose Cartesian equation is $4x^2 - 6xy + 9y^2 = 36$

$$4x^2 - 6xy + 9y^2 - 36 = 0 \quad \Rightarrow \quad A = 4,\ B = -6,\ C = 9$$

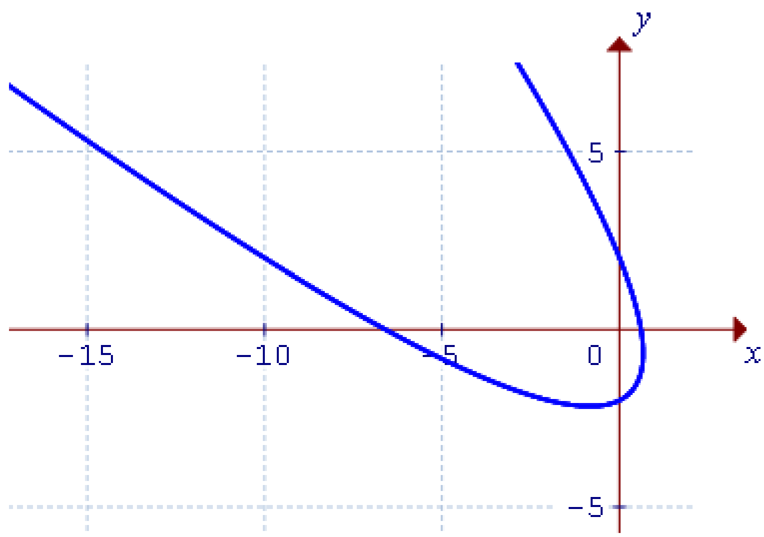


Example 3.2.2

Classify the conic section whose Cartesian equation is $3x^2 + 6xy + 3y^2 = 12$

Example 3.2.3

Classify the conic section whose Cartesian equation is $x^2 + 2xy + y^2 + 6x = 4$



Example 3.2.4

Classify the conic section whose Cartesian equation is $x^2 + 3xy + 2y^2 - x - 2y - 1 = 0$

$x^2 + 3xy + 2y^2 - x - 2y - 1 = 0 \quad \Rightarrow \quad A = 1,\ B = 3,\ C = 2$

$\Rightarrow \quad \Delta = B^2 - 4AC = 9 - 4 \times 1 \times 2 = 9 - 8 = 1 > 0$

The equation does have real solutions:

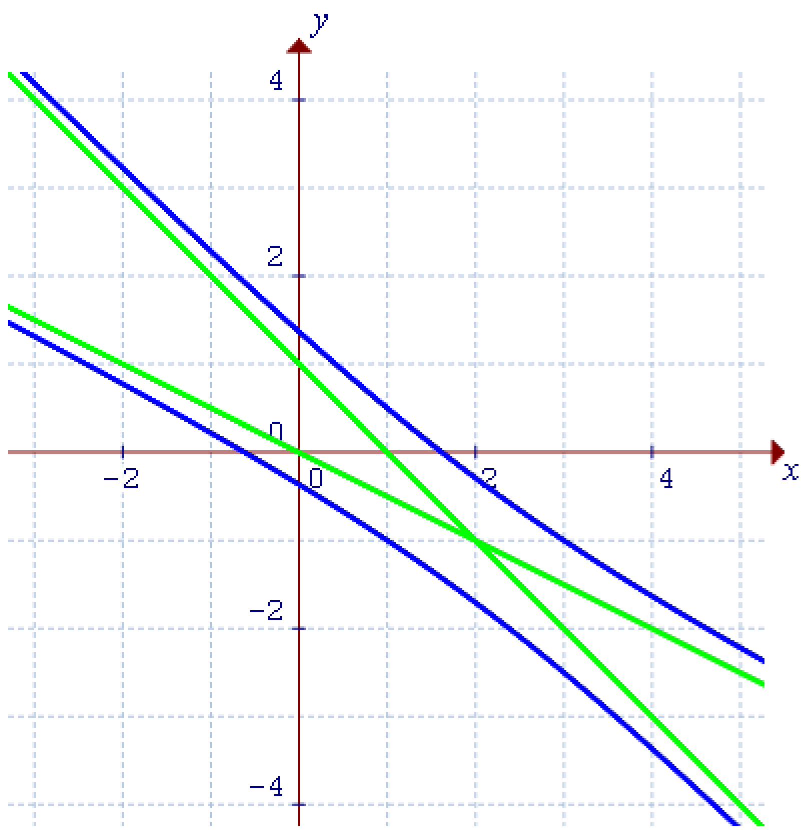
$y = 0 \quad \Rightarrow \quad x^2 - x - 1 = 0 \quad \Rightarrow \quad x = \dfrac{+1 \pm \sqrt{5}}{2}$ and

$x = 0 \quad \Rightarrow \quad 2y^2 - 2y - 1 = 0 \quad \Rightarrow \quad y = \dfrac{+1 \pm \sqrt{3}}{2}$

It is harder to show that this conic section is non-degenerate.

$\Delta > 0 \quad \Rightarrow$ **hyperbola**

The centre of the hyperbola is at $(2, -1)$ and the asymptotes are $y = 1 - x$ and $y = -\frac{1}{2}x$



3.3 <u>Polar Form for Conic Sections</u> [not examinable in ENGI 3425]

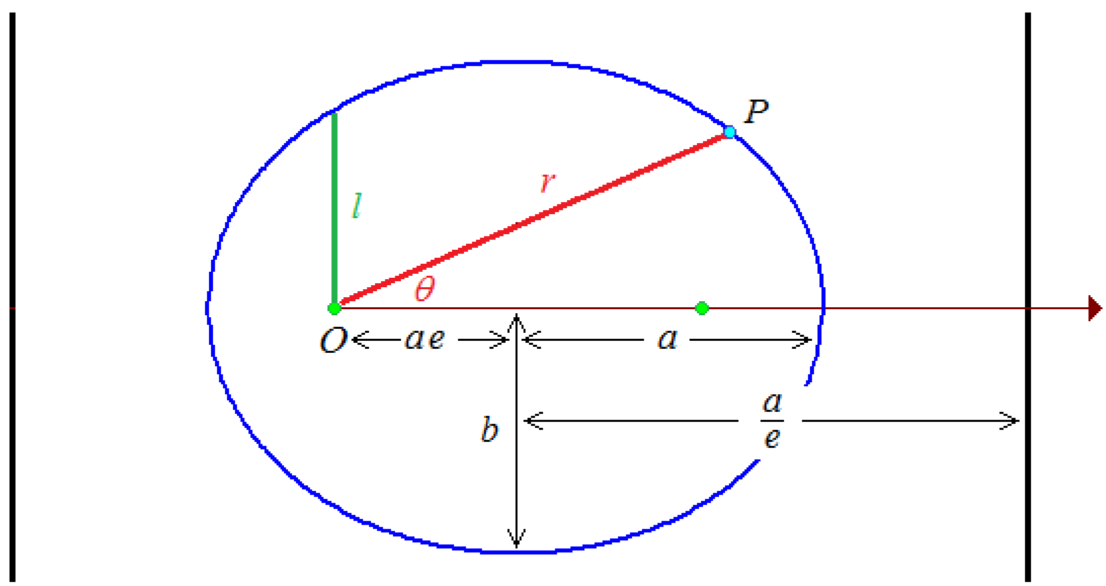
Conic sections may also be described compactly in polar coordinates, with one focus placed at the pole:

$$\boxed{r = \frac{l}{1 - e\cos\theta}}$$

where, for ellipses and hyperbolas, $l = \dfrac{b^2}{a}$ is the "semi latus rectum", which is the distance, parallel to the minor axis and the directrix, between the focus and the curve. In the case of the parabola, $l = 2a$.

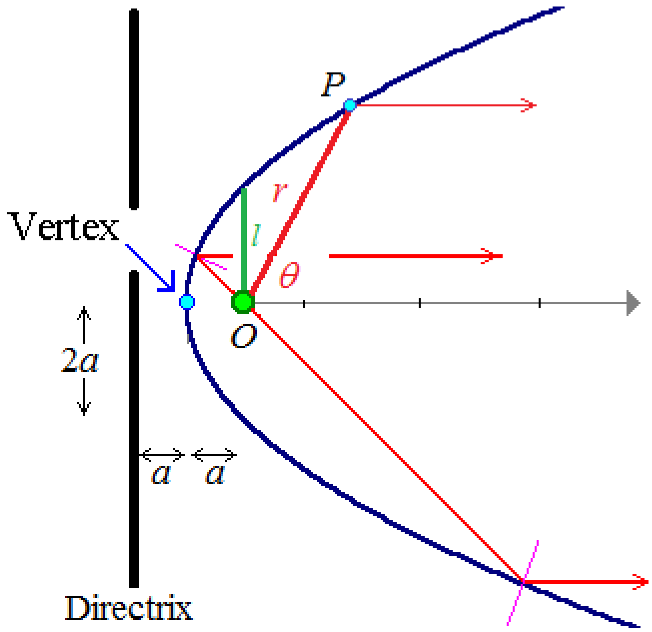
In some applications this polar form is preferred to the standard Cartesian form. The analysis of the motion of a particle due to an inverse-square-law-force (such as gravity), whose source is at one focus, is such an application.

Ellipse

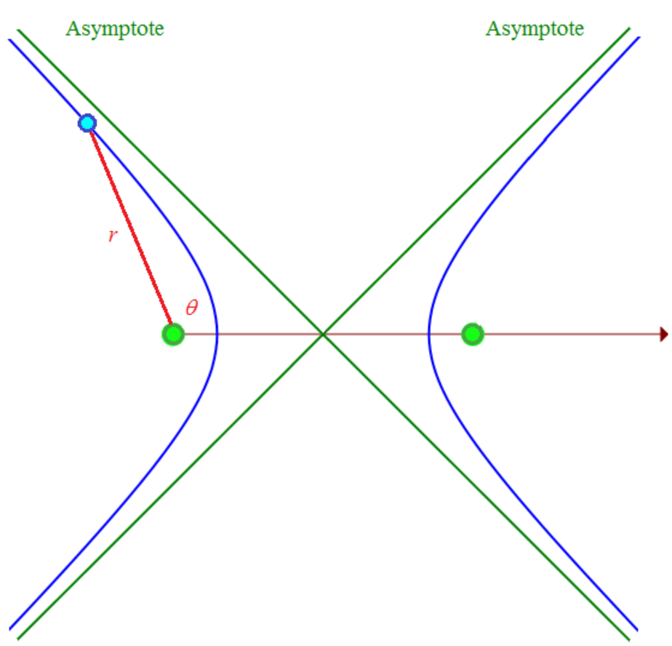


Parabola

$$r = \frac{l}{1 - \cos\theta}$$



Hyperbola



On the right branch of the hyperbola $r < 0$.

Moving up to three dimensions, we have the family of quadric surfaces (Chapter 4).

End of Chapter 3