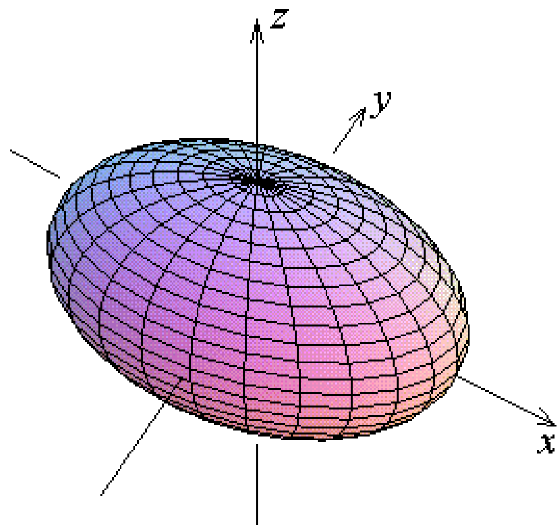
4.1 Classification of Quadric Surfaces

We shall consider only the simplest cases, where any planes of symmetry are located on the Cartesian coordinate planes. In nearly all cases, this eliminates "cross-product terms", such as *xy*, from the Cartesian equation of a surface. Except for the paraboloids, the centre is at the origin and the Cartesian equations involve only x^2, y^2, z^2 and constants.

The five main types of quadric surface are:

The **ellipsoid** (axis lengths a, b, c)

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1$$



The axis intercepts are at $(\pm a,0,0), (0,\pm b,0)$ and $(0,0,\pm c)$.

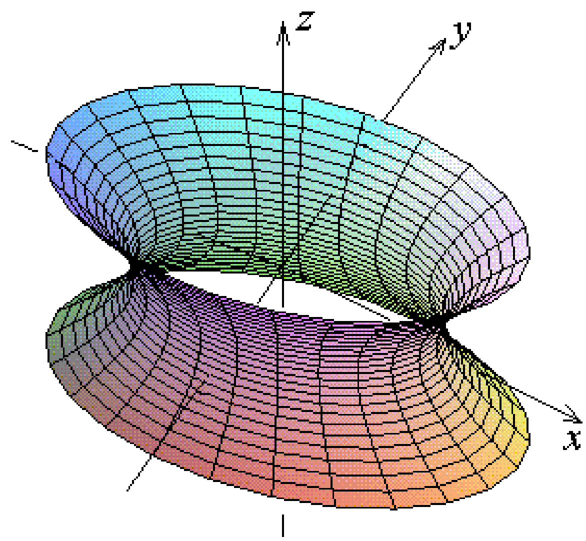
All three coordinate planes are planes of symmetry.

The cross-sections in the three coordinate planes are all ellipses.

Special cases (which are surfaces of revolution):
$a = b > c$: oblate spheroid (a "squashed sphere")
$a = b < c$: prolate spheroid (a "stretched sphere" or cigar shape)
$a = b = c$: sphere

Hyperboloid of One Sheet (Ellipse axis lengths a , b ; aligned along the z axis)

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2} = 1$$



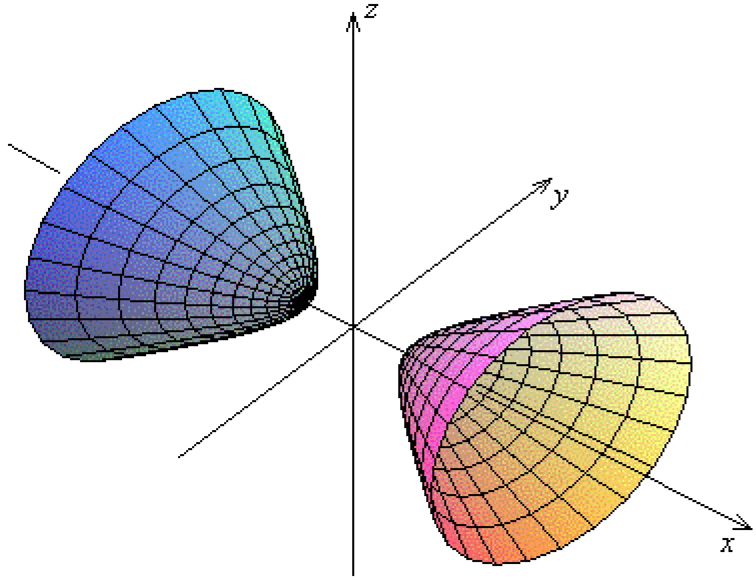
For hyperboloids, the central axis is associated with the "odd sign out".

In the case illustrated, the hyperboloid is aligned along the *z* axis.

The axis intercepts are at $(\pm a,0,0)$ and $(0,\pm b,0)$.
The vertical cross sections in the *x*-*z* and *y*-*z* planes are hyperbolae.
All horizontal cross sections are ellipses.

Hyperboloid of Two Sheets (Ellipse axis lengths b, c; aligned along the x axis)

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$



For hyperboloids, the central axis is associated with the "odd sign out".

In the case illustrated, the hyperboloid is aligned along the x axis.

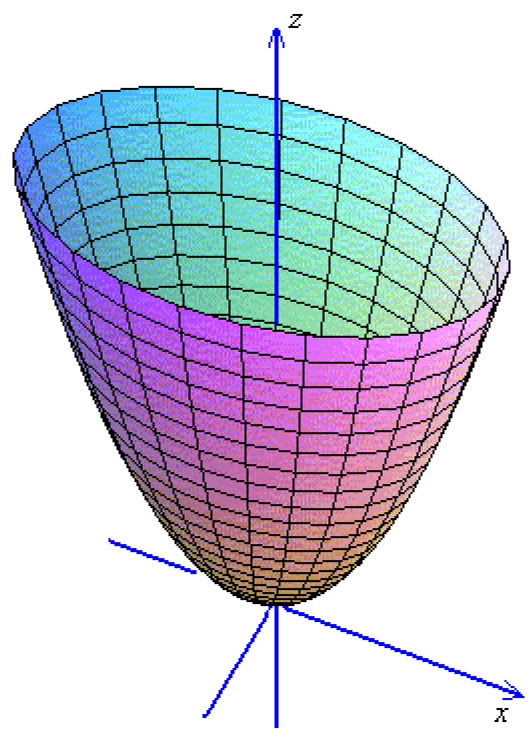
The axis intercepts are at $(\pm a, 0, 0)$ only.

Vertical cross sections parallel to the y-z plane are either ellipses or null.

All cross sections containing the x axis are hyperbolae.

Elliptic Paraboloid
(Ellipse axis lengths a, b;
aligned along the z axis)

$$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$



For paraboloids, the central axis is associated with the "odd exponent out".

In the case illustrated, the paraboloid is aligned along the z axis.

The only axis intercept is at the origin.

The vertical cross sections in the x-z and y-z planes are parabolae.
All horizontal cross sections are ellipses (for $z > 0$).

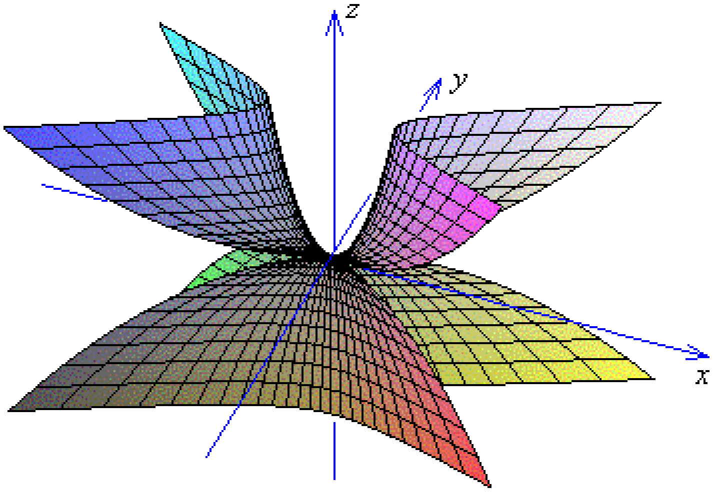
Hyperbolic Paraboloid (Hyperbola axis length a or b; aligned along the z axis)

$$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

For paraboloids, the central axis is associated with the "odd exponent out".

In the case illustrated, the paraboloid is aligned along the z axis.

The only axis intercept is at the origin.



The vertical cross section in the *x*-*z* plane is an upward-opening parabola.
The vertical cross section in the *y*-*z* plane is a downward-opening parabola.
All horizontal cross sections are hyperbolae, (except for a point at $z = 0$).

The plots of the five standard quadric surfaces shown here were generated in the software package Maple. The Maple worksheet is available from a link at "`http://www.engr.mun.ca/~ggeorge/3425/demos/index.html`".

Degenerate Cases:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 0 \quad :$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0 \quad :$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = -1 \quad :$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad :$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad :$$

$$\frac{y}{b} = \frac{x^2}{a^2} \quad :$$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 0$:

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = -1$:

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$:

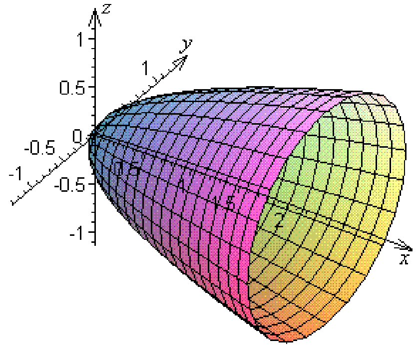
$\frac{x^2}{a^2} = 1$:

$\frac{x^2}{a^2} = 0$:

$\frac{x^2}{a^2} = -1$:

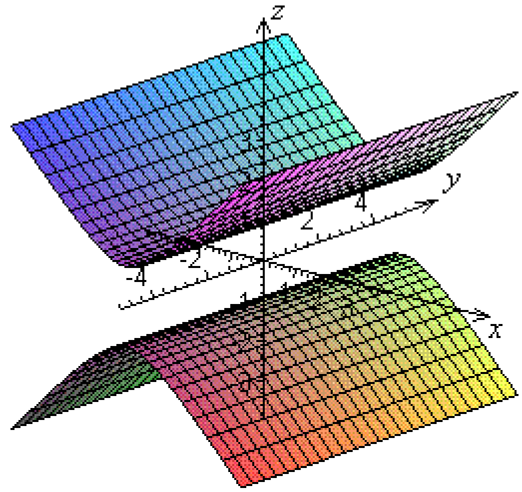
Example 4.1

Classify the quadric surface, whose Cartesian equation is $2x = 3y^2 + 4z^2$.



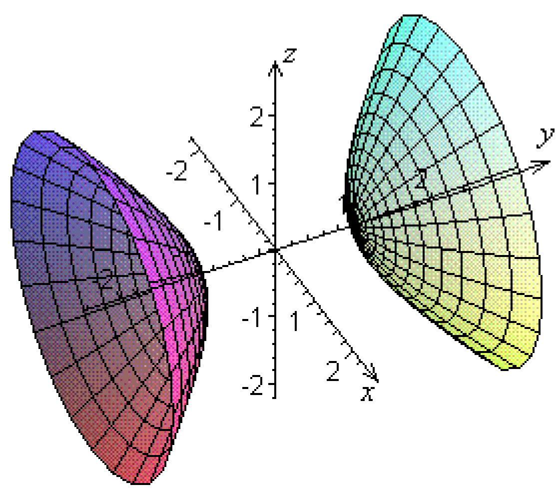
<u>Example 4.2</u>

Classify the quadric surface, whose Cartesian equation is $z^2 = 1 + x^2$.



<u>Example 4.3</u>

Classify the quadric surface, whose Cartesian equation is $x^2 - y^2 + z^2 + 1 = 0$.



More examples are in the problem sets.

[Space for additional notes]

[End of Chapter 4]