# 5. Parametric Vector Functions

**Contents:**

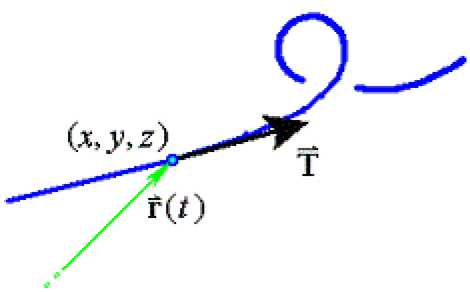


---

Note that any non-zero vector $\vec{\mathbf{r}}$ can be decomposed into its magnitude $r$ and its direction:

$$\vec{\mathbf{r}} \equiv r\hat{\mathbf{r}} \,, \quad \text{where } r \equiv |\vec{\mathbf{r}}| > 0$$

**Tangent Vector:**




$$\vec{\mathbf{T}} = \left[ \frac{dx}{dt} \;\; \frac{dy}{dt} \;\; \frac{dz}{dt} \right]^{\mathrm{T}} = \frac{d\vec{\mathbf{r}}}{dt}$$

If the parameter $t$ is time, then the tangent vector is also the **velocity vector**, $\vec{\mathbf{v}} = \dfrac{d\vec{\mathbf{r}}}{dt}$,

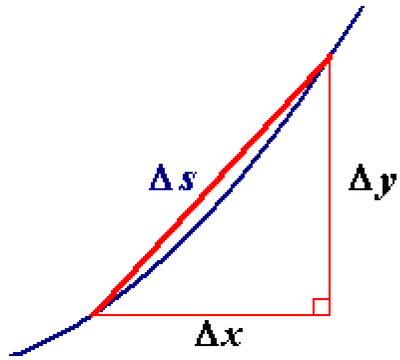
whose magnitude is the **speed** $v = \left| \dfrac{d\vec{\mathbf{r}}}{dt} \right|$.

The unit tangent is

$$\boxed{\mathbf{T} = \frac{d\vec{\mathbf{r}}}{dt} \div \left| \frac{d\vec{\mathbf{r}}}{dt} \right|}$$

---

## 5.1 Arc Length




In $\mathbb{R}^2$:

In $\mathbb{R}^3$:

The vector $\frac{d\vec{\mathbf{r}}}{dt}$ points in the direction of the tangent $\vec{\mathbf{T}}$ to the curve defined parametrically by $\vec{\mathbf{r}} = \vec{\mathbf{r}}(t)$.

<u>Example 5.1.1</u>

(a) Find the arc length along the curve defined by
$\vec{\mathbf{r}}(t) = \begin{bmatrix} 9t^2 & 12t^2 & 10t^3 \end{bmatrix}^{\mathrm{T}}$, from the origin to the point (9, 12, 10).

(b) Find the unit tangent $\mathbf{T}$.

(c) What happens to $\mathbf{T}$ as the curve passes through the origin?

---

**Arc Length for a Polar Curve**

For a polar curve defined by $r = f(\theta)$, $(\alpha \le \theta \le \beta)$, the parameter is $\theta$ and
$x = f(\theta)\cos\theta \qquad y = f(\theta)\sin\theta$.
Using the abbreviations $r = f(\theta)$, $r' = f'(\theta)$, $c = \cos\theta$, $s = \sin\theta$,

$\dfrac{dx}{d\theta} =$ and $\dfrac{dy}{d\theta} =$

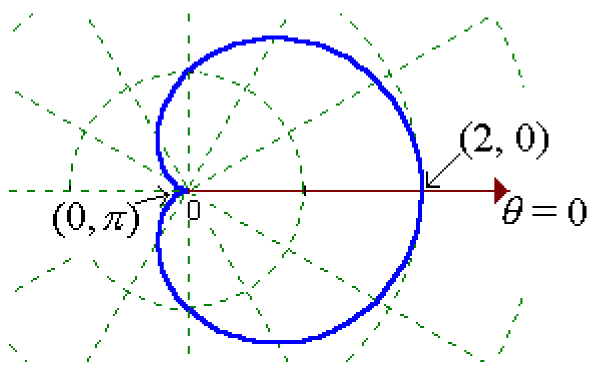
$$\Rightarrow \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 =$$

Therefore the arc length $L$ along the polar curve $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$ is

---

Example 5.1.2

Find the length $L$ of the perimeter of the cardioid $r = 1 + \cos\theta$



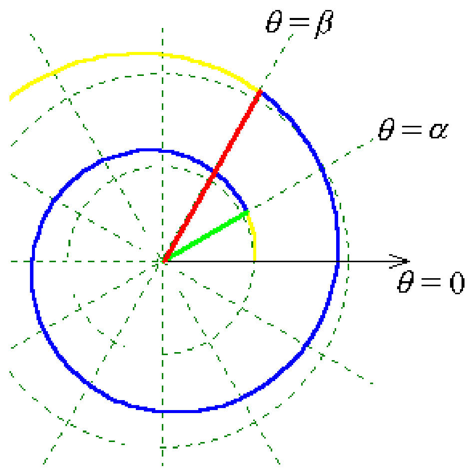
Example 5.1.2 (continued)

---

Example 5.1.3

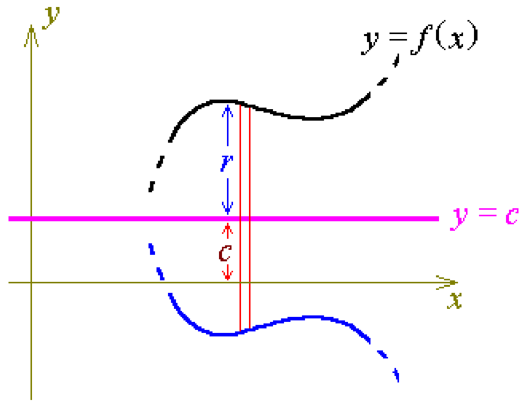
Find the arc length along the spiral curve $r = a\,e^{\theta}$ $\left(a > 0\right)$, from $\theta = \alpha$ to $\theta = \beta$.

---



## 5.2 Surfaces of Revolution

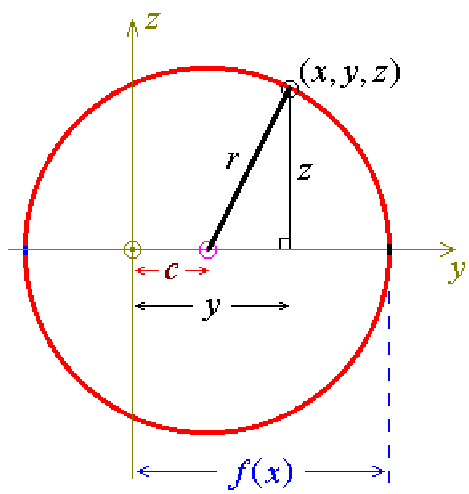
Consider a curve in the $x$-$y$ plane, defined by the equation $y = f(x)$.
If it is swept once around the line $y = c$, then it will generate a surface of revolution.




At any particular value of $x$, a thin cross-section through that surface, parallel to the $y$-$z$ plane, will be a circular disc of radius $r$, where

$$r =$$

Let us now view the circular disc face-on, (so that the $x$ axis and the axis of rotation are both pointing directly out of the page and the page is parallel to the $y$-$z$ plane).




Let $(x, y, z)$ be a general point on the surface of revolution.

From this diagram, one can see that

$$r^2 =$$

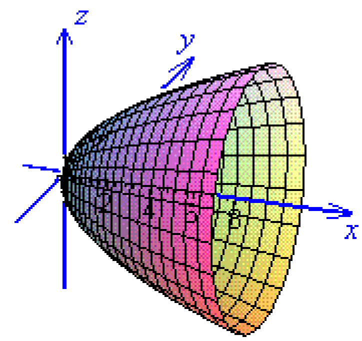
Therefore, the equation of the surface generated, when the curve $y = f(x)$ is rotated once around the axis $y = c$, is

Special case: When the curve $y = f(x)$ is rotated once around the $x$ axis, the equation of the surface of revolution is

Example 5.2.1

Find the equation of the surface generated, when the parabola $y^2 = 4ax$ is rotated once around the $x$ axis.

---

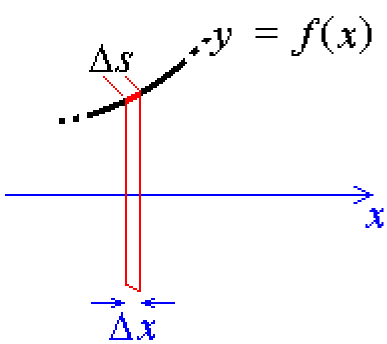



A Maple worksheet for this surface is available from the demonstration files section of the ENGI 3425 web site.

---

**The Curved Surface Area of a Surface of Revolution**

For a rotation around the $x$ axis,




the curved surface area swept out by the element of arc length $\Delta s$ is approximately the product of the circumference of a circle of radius $y$ with the length $\Delta s$.

Integrating along a section of the curve $y = f(x)$ from $x = a$ to $x = b$, the total curved surface area is

For a rotation of $y = f(x)$ about the axis $y = c$, the curved surface area is

Example 5.2.2

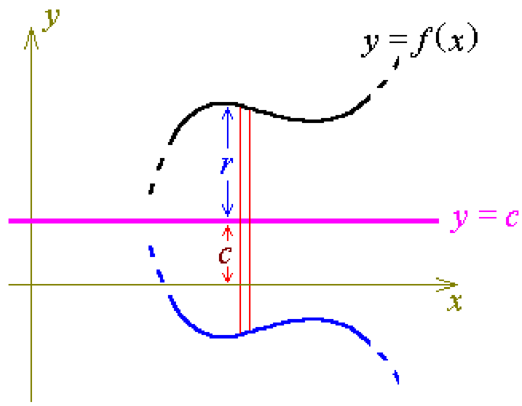
Find the curved surface area of the circular paraboloid generated by rotating the portion of the parabola $y^2 = 4cx$ $(c > 0)$ from $x = a$ $(\geq 0)$ to $x = b$ about the $x$ axis.

---

$$A = 2\pi \int_{x=a}^{x=b} |y| \, ds$$

**The Volume enclosed by a Surface of Revolution**

As noted above, a thin slice through a surface of revolution, at right angles to the axis of rotation, is approximately a circular disc of radius $r = | f(x) - c |$ and thickness $\Delta x$.




The volume of this very short circular cylinder is

Summing over all such elementary slices from $x = a$ to $x = b$, the total volume enclosed by the surface of revolution is

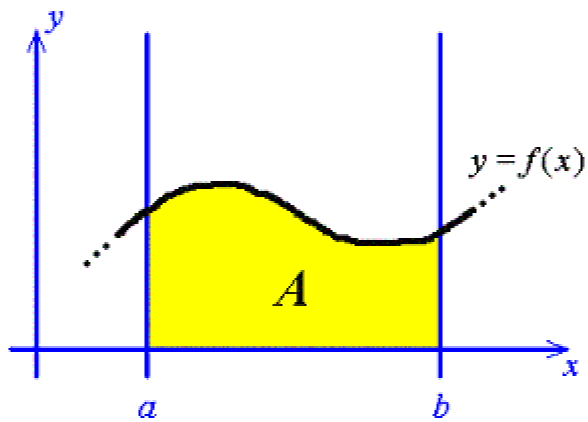
---

Example 5.2.3

Find the volume enclosed by the circular paraboloid generated by rotating the portion of the parabola $y^2 = 4cx$ $(c > 0)$ between $x = a$ $(\geq 0)$ and $x = b$ about the $x$ axis.

---

## 5.3 Area under a Parametric Curve $(x, y) = (x(t), y(t))$




$$A = \int_a^b f(x)\,dx$$

With parameterization $(x, y) = (x(t), y(t))$:

$$\boxed{A = \int_{t_a}^{t_b} |\,y\,| \frac{dx}{dt}\,dt}$$

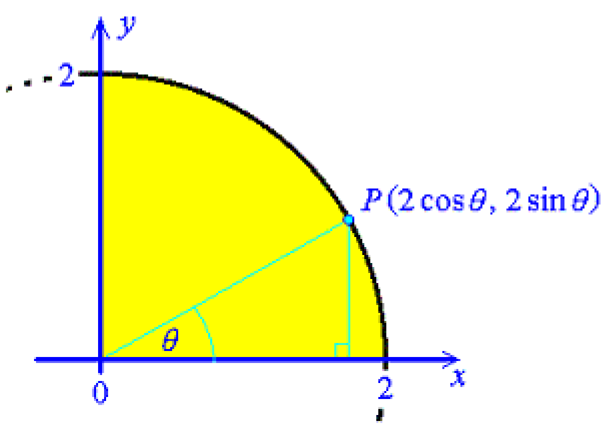
where $x(t_a) = a,\quad x(t_b) = b$ and $a < b$. Note that this does *not* guarantee $t_a < t_b$.

---

Example 5.3.1

Find the area enclosed in the first quadrant by the circle $x^2 + y^2 = 4$.

---

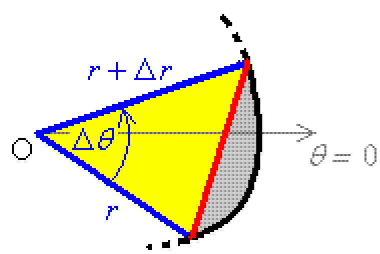


Example 5.3.1 (continued)

---

**Area Swept Out by a Polar Curve $r = f(\theta)$**

$\Delta A \approx$ Area of triangle



Example 5.3.2

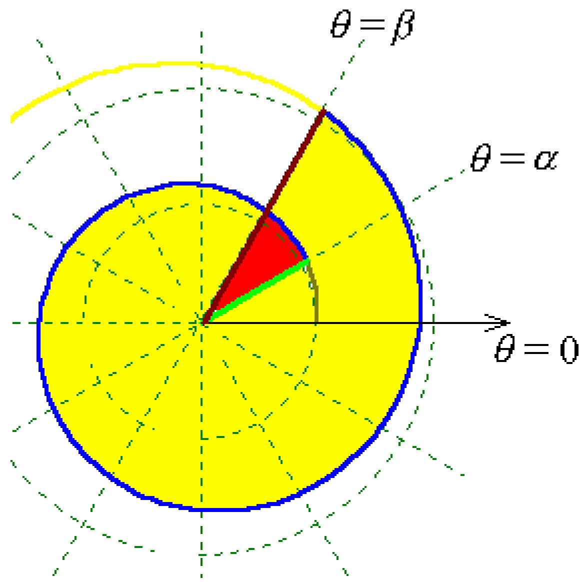
Find the area of a circular sector, radius $r$, angle $\theta$.

Example 5.3.3

Find the area swept out by the polar curve $r = a\,e^{\theta}$ over $\alpha < \theta < \beta$,
(where $a > 0$ and $\alpha < \beta < \alpha + 2\pi$).

---

The condition ($\alpha < \beta < \alpha + 2\pi$) prevents the same area being swept out more than once.




---

In general, the area bounded by two polar curves $r = f(\theta)$ and $r = g(\theta)$ and the radius vectors $\theta = \alpha$ and $\theta = \beta$ is

$$A = \frac{1}{2}\int_{\alpha}^{\beta}\left(\left(f(\theta)\right)^2 - \left(g(\theta)\right)^2\right)d\theta$$

[End of Chapter 5]