ENGI 4421 Probability and Statistics
Faculty of Engineering and Applied Science

# Problem Set 2

**[Introduction to Probability; odds, Venn diagrams, decision trees]**

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1. Draw a decision tree, involving choice and chance forks, for each of
   (a) a personal decision (for example, renewing a mortgage - what term?).
   (b) an engineering problem.

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2. An engineer states that the odds of a reinforced girder surviving an average load of 2 MN for 25 years is “7 to 1 on”. What is the engineer’s probability for this event?

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3. The event $E$ is defined to be
   “the score on a roll of a fair die is an odd prime number”;
   (note: the smallest prime number is 2).
   Calculate the odds on $E$, expressed as a ratio reduced to its lowest terms.

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4. For the sample space $\boldsymbol{S}$ which consists of the first ten natural numbers { 1, 2, ... , 10 }, events $A$, $B$, $C$ are defined as
   $A$ = { odd numbers in $\boldsymbol{S}$ }
   $B$ = { perfect square numbers in $\boldsymbol{S}$ }
   $C$ = { 3, 5 }
   [A “perfect square” is a whole number whose square root is also a whole number.]

   (a) Represent the above sets in a Venn diagram.
   (b) Express the set $D$ = { odd perfect square numbers in $\boldsymbol{S}$ } as a set function of the above sets. How many elements are in this set?
   (c) Simplify the expression $A \cap C$ .
   (d) Simplify the expression $A \cup C$ .
   (e) Simplify the expression $B \cap C$ .
   (f) What elements are in the set $\sim A \cap \sim B \cap \sim C$ ?

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5. In the quality control of mass-produced ceramic tiles,
the probability that the next tile checked by an inspector has an air bubble is **.**05000,
the probability that it is below acceptable mass is **.**04000,
the probability that it has a crack is **.**02000,
the probability that it either has an air bubble or is below acceptable mass or both is **.**08800,
the probability that it is either below acceptable mass or cracked or both is **.**05880,
the probability that it either is cracked or has an air bubble or both is **.**06800 and
the probability that it has at least one of these imperfections is **.**10484.

(a) What is the probability that the next tile checked by the inspector has **none** of these imperfections?
(b) What is the probability that the next tile checked by the inspector has an air bubble as its **only** imperfection?
(c) What is the probability that the next tile checked by the inspector has an air bubble **and** is cracked?
(d) What is the probability that the next tile checked by the inspector has **all** of these
(e) Are these three types of imperfection mutually exclusive?
(f) Are these three types of imperfection [stochastically] independent?

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6. A leading engineering company presently has bids out on three projects.
Let $A$ = { project 1 is awarded to the company },
$B$ = { project 2 is awarded to the company }
and $C$ = { project 3 is awarded to the company }
and suppose that

$$\mathrm{P}[A] = .86, \quad \mathrm{P}[B] = .79, \quad \mathrm{P}[C] = .58,$$

$$\mathrm{P}[A \wedge B] = .70, \quad \mathrm{P}[B \wedge C] = .49, \quad \mathrm{P}[C \wedge A] = .50$$

and $\mathrm{P}[A \wedge B \wedge C] = .45$.

(a) Draw a Venn diagram that represents this situation.
(b) Find the probability that the company wins **exactly one** of the three bids.

Express in words each of the following events **and** compute the probability of each event:

(c) $\tilde{A} \wedge \tilde{B}$
(d) $A \vee B \vee C$
(e) $\tilde{A} \wedge \tilde{B} \wedge \tilde{C}$
(f) $A \wedge B \wedge \tilde{C}$
(g) $\tilde{A} \wedge \tilde{B} \wedge C$

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7. A factory manager is trying to decide whether to invest in a new quality control process on a production line. If the manager decides against the investment, then a decision to accept or reject a product will be based only on her prior assessment of the probability that the product is defective, together with the consequences of each possible outcome:

reject defective product (true positive),
reject good product (false positive),
accept defective product (false negative),
accept good product (true negative).

If the manager decides to invest in the new quality control process, then a decision to accept or reject a product will be based on the verdict of the quality control process, together with the consequences of each possible outcome and the cost of the investment.

Draw a decision tree diagram that illustrates this situation.

[Recall that decision nodes should be represented by squares and chance nodes by circles. Be careful to place the nodes in the correct order. We will develop this example further in Problem Set 4.]

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8. *Bonus Question*: Odds vs. Bookmaker's Odds
Five teams are competing in a design competition which exactly one of them will win. An observer places a non-refundable deposit of $1 for the opportunity of receiving a reward for a correct prediction of which team will win. The reward is based upon the odds quoted here:

| | |
|---|---|
| Team $A$ : | **5:3 on** |
| Team $B$ : | **13:7 against** |
| Team $C$ : | **4:1 against** |
| Team $D$ : | **19:1 against** |
| Team $E$ : | **39:1 against** |

(a) Find the probabilities associated with the odds quoted above.
(b) Show that the probabilities associated with these odds are *not* coherent.
(c) For each event, find the reward for a correct prediction (that that team will win).
(d) If 25 observers pay a deposit and predict that team $A$ will win, 14 predict $B$, 8 predict $C$, 2 predict $D$ and 1 predicts $E$, then find the bookmaker's profit for each of the five outcomes.
(e) Now assume that the relative chances of victory for each team are reflected by the relative odds quoted above, (so that, for example, team $C$ really is four times as likely to win as team $D$). Rescale the five odds quoted above so that the bet becomes a fair bet.
(f) Under this fair bet, what now is the bookmaker's profit for each of the five outcomes?

8 (g) Comment on the relationship between the distribution of profits and the distribution of the numbers of deposits.

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A general approach to this bonus question follows:

An operator accepts deposits on a partition $\{E_i\}$.
$k_i$ people each place a non-refundable deposit $p_i s_i$ on $E_i$ occurring.
Note that $p_i$ is a measure of the likelihood of $E_i$ occurring.
(The more likely $E_i$ is, the greater the deposit that the operator will require).
If $E_i$ occurs, then the operator pays a stake $s_i$ to each of the $k_i$ contractors,
(but still retains all of the deposits).
If $E_i$ does not occur, then each of the $k_i$ contractors loses the deposit $p_i s_i$.

The operator's gain if $E_h$ is true is

**(gain) = (revenue) – (payout)**

$$g_h = \left( \sum_{i=1}^{n} k_i p_i s_i \right) - k_h s_h \qquad \text{(for } h = 1, 2, \ldots, n\text{)}$$

Now assume a common situation of equal deposits:

$$p_1 s_1 = p_2 s_2 = \ldots = p_n s_n = b$$

Then

$$g_h = \left( b \sum_{i=1}^{n} k_i \right) - k_h \left( \frac{b}{p_h} \right) \qquad \text{(for } h = 1, 2, \ldots, n\text{)}$$

The number $p_h$ is a measure of how likely the gain $g_h$ is to occur.
Therefore use $p_h$ as a weighting factor, to arrive at an expected gain:

$$\mathrm{E}[G] = \sum_{h=1}^{n} p_h g_h = b \sum_{h=1}^{n} p_h \left( \left( \sum_{i=1}^{n} k_i \right) - k_h \left( \frac{1}{p_h} \right) \right)$$

$$= b \left( \left( \sum_{i=1}^{n} k_i \right) \left( \sum_{h=1}^{n} p_h \right) - \left( \sum_{h=1}^{n} k_h \right) \right) = b \left( \sum_{i=1}^{n} k_i \right) \left( \left( \sum_{i=1}^{n} p_i \right) - 1 \right)$$

$\mathrm{E}[G] = 0$ if and only if $\sum_{i=1}^{n} p_i = 1$

A **fair bet** is then assured if $\boxed{\sum_{i=1}^{n} p_i = 1}$ (**total probability theorem**);
the probabilities are then **coherent**.

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