ENGI 4421 Probability and Statistics
Faculty of Engineering and Applied Science

Problem Set 5

Probability distributions, expectation, variance

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1. A discrete function of *x* is defined by

$$p(x) = \begin{cases} \frac{1}{8} & (x = -1, +1) \\ \frac{3}{4} & (x = 0) \\ 0 & (\text{otherwise}) \end{cases}$$

(a) Verify that $p(x)$ is a well-defined probability mass function (p.m.f.).
(b) Find the corresponding cumulative distribution function (c.d.f.), $F(x)$ **and** sketch its graph.
(c) Find the population mean μ and the population median $\tilde{\mu}$.
(d) Find the population standard deviation σ.

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2. The probability mass function for X = {the number of major defects in a randomly selected appliance of a certain type} is

x	0	1	2	3	4
***p*(*x*)**	.09	.29	.36	.21	.05

(a) Verify that $p(x)$ is a well-defined probability mass function.
(b) Find the corresponding cumulative distribution function, $F(x)$.
(c) Compute $E[X]$.
(d) Compute $V[X]$ directly from the definition.
(e) Compute $V[X]$ using the shortcut formula.
(f) Compute the standard deviation of *X*.

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3. A function $f(x)$ of the continuous quantity x is defined by

$$f(x) = \begin{cases} \frac{1}{36}\left(x^2 - 2x\right) & (0 \le x \le 6) \\ 0 & (\text{otherwise}) \end{cases}$$

Can $f(x)$ be a well-defined probability density function? State why or why not.

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4. [Navidi, Exercises 2.4, Question 18, modified]
The lifetime, in years, of a certain type of fuel cell is a random quantity with probability density function

$$f(x) = \frac{81}{(x+3)^4} \quad (x>0)$$

(a) Verify that $f(x)$ is a valid probability density function.
(b) Find the cumulative distribution function $F(x)$ of the lifetime.
(c) What is the probability that a fuel cell lasts between one and three years?
(d) Find the mean lifetime.
(e) Find the variance of the lifetimes.
(f) Find the median lifetime.
(g) Find the 30^{th} percentile of the lifetimes.

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5. A contractor knows that the probability that it will win a particular type of contract in a particular tender call is *p*. The value of *p* is the same from one tender call to the next.

(a) Find the probability that the first such contract won by the contractor will occur in the very first tender call.
(b) in the second tender call.
(c) in tender call number x.

The remainder of this question is of a more challenging nature.

(d) A random quantity X that has the probability distribution of part (c) of this question has a geometric probability distribution with parameter *p*.
Find the cumulative distribution function (c.d.f.) $F(x)$
[Hint: you may quote the formula for the n^{th} partial sum of a geometric series with first term *a* and common ratio *r*:

$$s_n = \sum_{k=1}^{n} a r^{k-1} = \frac{a(1-r^n)}{1-r} \quad]$$

(e) Find the mode *m* of *X*.
[The mode is the value of *x* at which the p.m.f. $p(x)$ achieves its greatest value – The mode is the most common or, literally, the most “fashionable” value.]
(f) Find the median value $\tilde{\mu}$ of *X*.
[Note that the median is defined by $\mathrm{P}\left[X < \tilde{\mu}\right] = \frac{1}{2}$]
(g) Show that the mean value of X is $\mu = \mathrm{E}[X] = 1/p$.
(h) Evaluate the mode, median and mean in the case $p = .25$.

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6. The cumulative distribution function $F(x)$ for a continuous random quantity X is

$$F(x) = \begin{cases} +\tfrac{1}{2}e^{3x} & (x < 0) \\ 1 - \tfrac{1}{2}e^{-3x} & (x \geq 0) \end{cases}$$

(a) Find the probability density function $f(x)$ and express it in its simplest form, (in a single-line definition that is valid for all values of x).
(b) Find the median value $\tilde{\mu}$. [Note that $F(\tilde{\mu}) = 1/2$]
(c) Find the mode.
(d) Evaluate $\text{P}[\,|X| < 0.1]$.

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7. The probability density function (p.d.f.) of a ***continuous*** random quantity X is given by

$$f(x) = kx^4(1-x)^2 = k\left(x^4 - 2x^5 + x^6\right) \qquad (0 \leq x \leq 1)$$

(a) Prove that $f(x)$ is a probability density function if and only if $k = 105$.
(b) Find the cumulative distribution function (c.d.f.) $F(x)$ for X.
(c) Correct to three decimal places, find $\text{P}[X \leq 0.8]$.
(d) Find $\text{P}[X \leq 1.2]$.
(e) Find the exact value of $\text{E}[X]$.

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8. The Cauchy probability density function with parameter a,

$$f(x; a) = \frac{k}{x^2 + a^2}, \qquad (a > 0)$$

resembles, at first sight, the bell shaped Normal curve, but with much thicker tails.
(a) Find the value that k must have in order for $f(x; a)$ to be a well defined probability density function.
[You may quote the identity $\displaystyle\int \frac{dx}{x^2 + a^2} = \frac{1}{a}\text{Arctan}\left(\frac{x}{a}\right) + C$
(which can be established using the trigonometric substitution $x = a \tan\theta$ and the identities
$\displaystyle\frac{d}{d\theta}(\tan\theta) = \sec^2\theta = 1 + \tan^2\theta$).]
(b) Find the cumulative distribution function $F(x; a)$ for the Cauchy distribution.
(c) Find the "inter-quartile range" *IQR*, (which is the distance between the values of the quartiles x_L and x_U, at which $F(x_L; a) = \tfrac{1}{4}$ and $F(x_U; a) = \tfrac{3}{4}$ respectively).
(d) Find $\mu = \text{E}[X]$.
(e) Find the standard deviation σ.

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