Beta and Gamma Distributions

Obtaining the parameters from the mean and standard deviation

Beta Distribution (with $A < X < B$):

The population mean and variance are

$$\mu = A + (B-A)\frac{\alpha}{\alpha+\beta} \quad \text{and} \quad \sigma^2 = \frac{(B-A)^2\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} \quad (\alpha>0,\ \beta>0)$$

from which

$$\alpha = (\mu-A)\cdot\frac{(B-\mu)(\mu-A)-\sigma^2}{(B-A)\sigma^2} \quad \text{and} \quad \beta = (B-\mu)\cdot\frac{(B-\mu)(\mu-A)-\sigma^2}{(B-A)\sigma^2}$$

with constraints $0<A<\mu<B$ and $0<\sigma<\sqrt{(B-\mu)(\mu-A)}$

Gamma Distribution (with $X > 0$):

The population mean and variance are

$$\mu = \alpha\beta \quad \text{and} \quad \sigma^2 = \alpha\beta^2 \quad (\alpha>0,\ \beta>0)$$

from which

$$\alpha = \frac{\mu^2}{\sigma^2} \quad \text{and} \quad \beta = \frac{\sigma^2}{\mu} \quad (\mu>0,\ \sigma>0)$$

The **proof for the Gamma distribution** is straightforward:

$$\frac{\mu^2}{\sigma^2} = \frac{\alpha^2\beta^2}{\alpha\beta^2} = \alpha \quad \text{and} \quad \frac{\sigma^2}{\mu} = \frac{\alpha\beta^2}{\alpha\beta} = \beta$$

Proof for the Beta distribution:

From the formula for the population mean μ in terms of A, B, α, β,

$$\mu = A + (B-A)\frac{\alpha}{\alpha+\beta} \;\Rightarrow\; \mu - A = (B-A)\frac{\alpha}{\alpha+\beta} \;\Rightarrow\; \frac{\alpha+\beta}{\alpha} = \frac{B-A}{\mu-A}$$

$$\Rightarrow\; \alpha+\beta = \frac{B-A}{\mu-A}\alpha, \qquad \alpha+\beta+1 = \frac{B-A}{\mu-A}\alpha + 1 = \frac{(B-A)\alpha + (\mu-A)}{\mu-A}$$

and $\beta = \left(\dfrac{B-A}{\mu-A} - 1\right)\alpha = \dfrac{(B-A)-(\mu-A)}{\mu-A}\alpha = \dfrac{B-\mu}{\mu-A}\alpha$

Substitute into the expression for the population variance:

$$\sigma^2 = \frac{(B-A)^2\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} \;\Rightarrow\; (\alpha+\beta)^2(\alpha+\beta+1)\sigma^2 = (B-A)^2\alpha\beta$$

$$\Rightarrow\; \left(\frac{B-A}{\mu-A}\alpha\right)^2\left(\frac{(B-A)\alpha + (\mu-A)}{\mu-A}\right)\sigma^2 = (B-A)^2\alpha\left(\frac{B-\mu}{\mu-A}\alpha\right)$$

$$\Rightarrow\; \cancel{\frac{(B-A)^2\alpha^2}{\mu-A}}\left(\frac{(B-A)\alpha + (\mu-A)}{(\mu-A)^2}\right)\sigma^2 = \cancel{\frac{(B-A)^2\alpha^2}{\mu-A}}(B-\mu)$$

$$\Rightarrow\; \big((B-A)\alpha + (\mu-A)\big)\sigma^2 = (B-\mu)(\mu-A)^2$$

$$\Rightarrow\; (B-A)\alpha\sigma^2 + (\mu-A)\sigma^2 = (B-\mu)(\mu-A)^2$$

$$\Rightarrow\; (B-A)\alpha\sigma^2 = (B-\mu)(\mu-A)^2 - (\mu-A)\sigma^2$$

$$\Rightarrow\; (B-A)\alpha\sigma^2 = (\mu-A)\big((B-\mu)(\mu-A) - \sigma^2\big)$$

$$\Rightarrow\; \alpha = \frac{(\mu-A)\big((B-\mu)(\mu-A) - \sigma^2\big)}{(B-A)\sigma^2}$$

from which

$$\beta = \frac{B-\mu}{\mu-A}\alpha = \frac{B-\mu}{\cancel{\mu-A}}\cdot\frac{\cancel{(\mu-A)}\big((B-\mu)(\mu-A) - \sigma^2\big)}{(B-A)\sigma^2}$$

Obviously $0 < A < X < B \;\Rightarrow\; 0 < A < \mu < B$ and

$\alpha > 0,\ \beta > 0 \;\Rightarrow\; (B-\mu)(\mu-A) - \sigma^2 > 0 \;\Rightarrow\; \sigma^2 < (B-\mu)(\mu-A)$

[Back to the index of demonstration files]