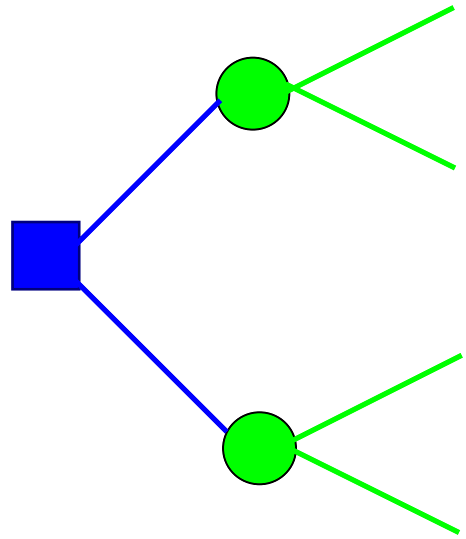
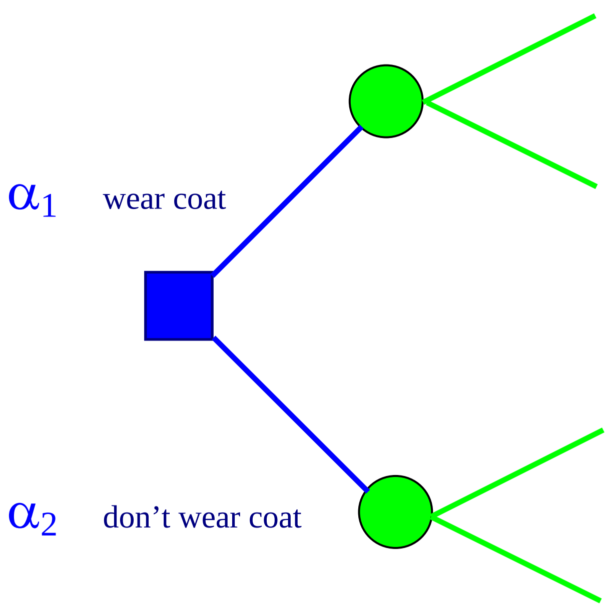
Probability

Decision trees

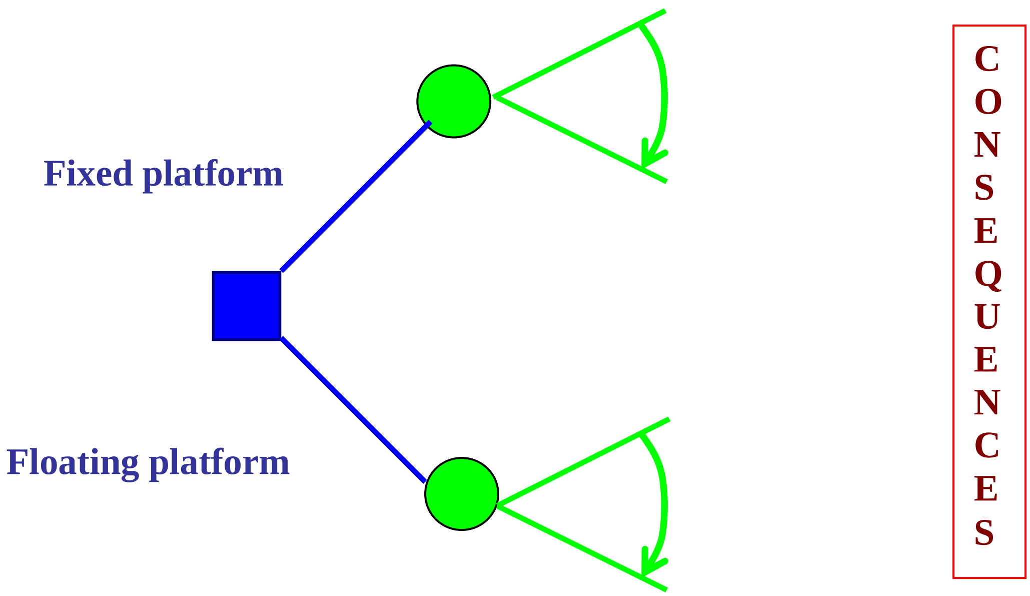


Example 2.01



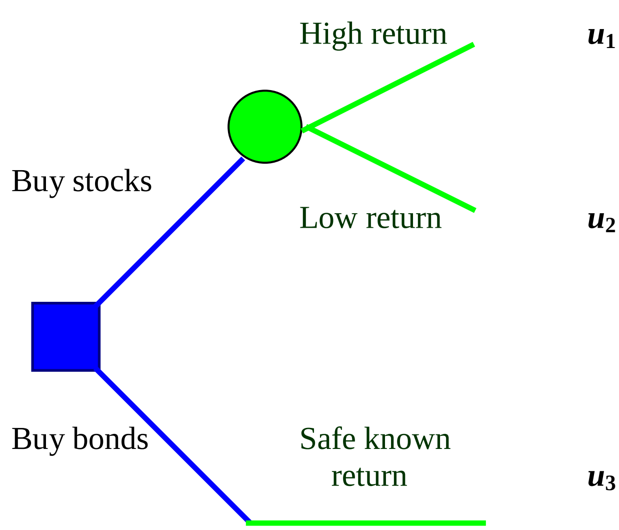
Example 2.02

(platform for oil & gas development)



Example 2.03

Investment



Fair bet

Example 2.04

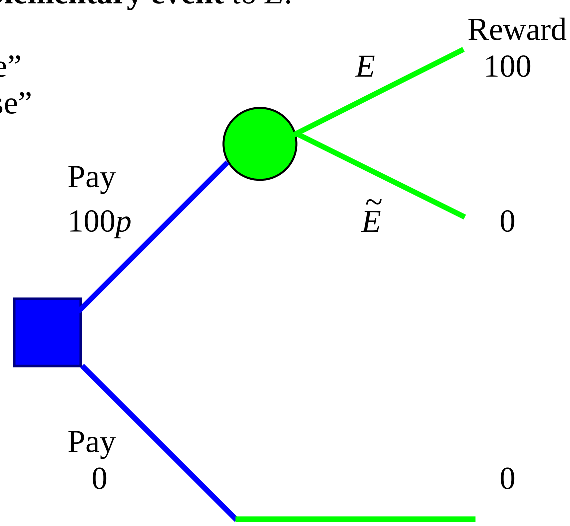
A client gives a \$100 reward iff (if and only if) the contractor's circuit board passes a reliability test. The contractor must pay a non-refundable deposit of \$100p with the bid. What is a fair price for the deposit?

Let E = (the event that the circuit board passes the test)
and $\tilde{E} = \sim E$ = not-E = (the event that the circuit board fails the test)
then $\tilde{E}$ is known as the **complementary event** to E.

Let $E = 1$ represent "E is true"
and $E = 0$ represent "E is false"
then

$$\boxed{E + \tilde{E} \ = \qquad}$$

Decision tree:



$$p = \frac{\text{deposit}}{\text{reward}}$$

If the contract is taken (= upper branches of decision tree):

(Gain if $\tilde{E}$) =

(Gain if E) =

Therefore Gain =

where E is random, (= 0 or 1; E is a **Bernoulli random quantity**).

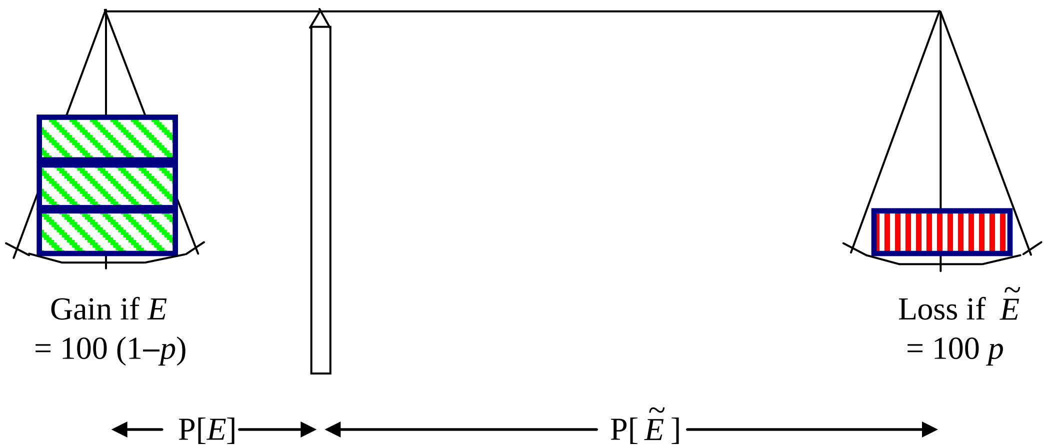
If the contract is not taken (= lowest branch of decision tree):

Gain $\equiv 0$

A fair bet $\Rightarrow$ **indifference** between decisions

$\Rightarrow$

<u>Balance of Judgement</u>:



The bet is fair iff gain and loss balance:

Taking moments: $100\,(1-p) \times \mathrm{P}[E] = 100\,p \times \mathrm{P}[\tilde{E}]$
But $\tilde{E} = 1 - E$ and $\mathrm{P}[\tilde{E}] = 1 - \mathrm{P}[E]$
$\Rightarrow (1 - p + p) \times \mathrm{P}[E] = p$

Therefore the fair price for the bid deposit occurs when $\boxed{p = \mathrm{P}[E]}$
and the fair price is (deposit) = (contract reward) $\times$ P[E].

<u>Example 2.04</u> (continued)

Suppose that past experience suggests that E occurs 24% of the time.
Then we estimate that P[E] = .24 and the fair bid is $100 \times .24 =$ **<u>$24</u>.**

<u>Odds</u>

Let s be the reward at stake in the contract (= $100 in example 2.04).
The odds on E occurring are the ratio r, where

$$r = \frac{\text{loss if } \tilde{E}}{\text{gain if } E} = \frac{s\,p}{s\,(1-p)} = \frac{P[E]}{P[\tilde{E}]} = \frac{p}{\tilde{p}} = \frac{p}{1-p} \quad \Rightarrow P[E] = \frac{r}{r+1}$$

In example 2.04,

$r =$

"Even odds" $\Rightarrow$

"Odds on" when "Odds against" when

<u>Coherence</u>:

Suppose that no more than one of the events $\{E_1, E_2, \ldots, E_n\}$ can occur. Then the events are **incompatible** (= **mutually exclusive**).

If the events $\{E_1, E_2, \ldots, E_n\}$ are such that they exhaust all possibilities, (so that at least one of them must occur), then the events are **exhaustive**.

If the events $\{E_1, E_2, \ldots, E_n\}$ are both incompatible and exhaustive, (so that *exactly one* of them must occur), then they form a **partition**, and

$$E_1 + E_2 + \ldots + E_n =$$

A set of probabilities $\{p_1, p_2, \ldots, p_n\}$ for a partition $\{E_1, E_2, \ldots, E_n\}$ is **coherent** if only if

[See the bonus question in Problem Set 2 for an exploration of this concept of coherence.]

<u>Notation</u>:
$A \wedge B =$ events A and B both occur; $A \wedge B = A \times B = AB$

$A \vee B =$ event A or B (or both) occurs;
(but $A \vee B \neq A + B$ unless A, B are incompatible)

<u>Some definitions</u>: [Navidi Section 2.1; Devore Sections 2.1-2.2]

Experiment = process leading to a single outcome

Sample point (= simple event) = one possible outcome (which precludes all other outcomes)

Event E = set of related sample points

Possibility Space = universal set = Sample Space S =

By the definition of S, any event E is a subset of S: $E \subseteq S$

<u>Classical definition of probability</u> (when sample points are equally likely):

$$\mathrm{P}[E] = \frac{n(E)}{n(S)} ,$$

where $n(E)$ = the number of [equally likely] sample points inside the event E.

More generally, the probability of an event E can be calculated as the sum of the probabilities of all of the sample points included in that event:

$$\boxed{\mathrm{P}[\boldsymbol{E}] = \Sigma\ \mathrm{P}[X]}$$

(summed over all sample points X in E.)

<u>Empirical definition of probability</u>:

$\mathrm{P}[E] =$

<u>Example 2.05</u> (illustrating the evolution of relative frequency with an ever increasing number of trials):

`http://www.engr.mun.ca/~ggeorge/4421/demos/Cointoss.xlsx`

or import the following macro into a MINITAB session:

`http://www.engr.mun.ca/~ggeorge/4421/demos/Coins.mac`

Example 2.06: rolling a standard fair die. The sample space is

$$S = \{ 1, 2, 3, 4, 5, 6 \}$$

$n(S) = 6$ (the sample points are equally likely)

$$P[1] = 1/6 = P[2] = P[3] = \ldots$$

$P[S] =$

The **empty set** (= null set) = $\boldsymbol{\emptyset} = \{\}$

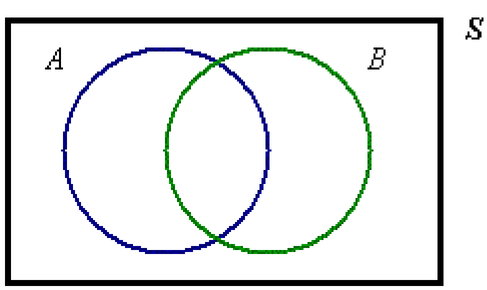
$P[\boldsymbol{\emptyset}] =$

The **complement** of a set A is A' (or $\tilde{A}$, A^*, A^c, NOT A, $\sim A$, $\bar{A}$).

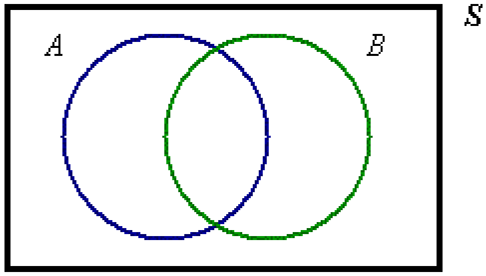
$n(\sim A) = n(S) - n(A)$ and

$$\boxed{P[\sim A] = 1 - P[A]}$$

The **union** $A \cup B$ = (A OR B) = $A \vee B$



The **intersection** $A \cap B = (A \text{ AND } B) = A \wedge B = A \times B = A\,B$



For *any* set or event **E** :

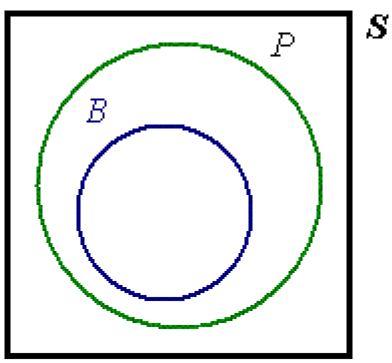
$\emptyset \cup \mathbf{E} =$ $\qquad$ $\mathbf{E} \cap \sim\mathbf{E} =$

$\emptyset \cap \mathbf{E} =$ $\qquad$ $\mathbf{E} \cup \sim\mathbf{E} =$

$S \cup \mathbf{E} =$ $\qquad$ $\sim(\sim\mathbf{E}) =$

$S \cap \mathbf{E} =$ $\qquad$ $\sim\emptyset =$

The set B is a **subset** of the set P : $\qquad B \subseteq P$.



If it is also true that $P \subseteq B$, then $P = B$ (the two sets are identical).

If $B \subseteq P$, $B \neq P$ and $B \neq \emptyset$, then $B \subset P$ (B is a **proper subset** of the set P).

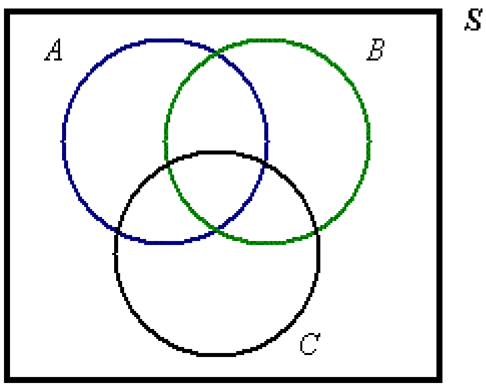
$B \cap P =$ $\qquad$ For *any* set or event **E** : $\qquad \emptyset \subseteq \mathbf{E} \subseteq S$

$B \cup P =$ $\qquad$ Also: $B \cap \sim P =$

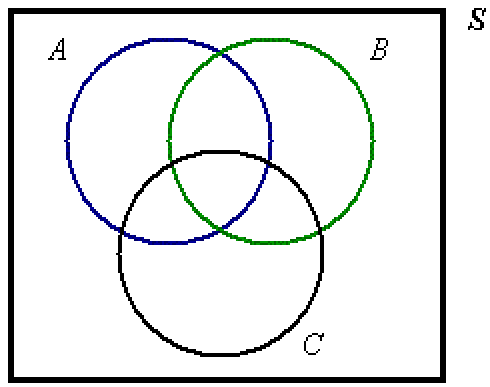
Example 2.07

Examples of Venn diagrams:

1. Events A and B both occur.



2. Event A occurs but event C does not.



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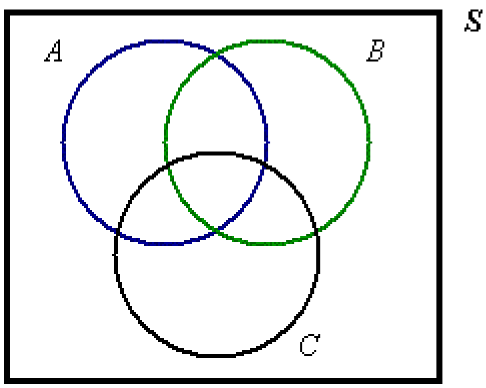
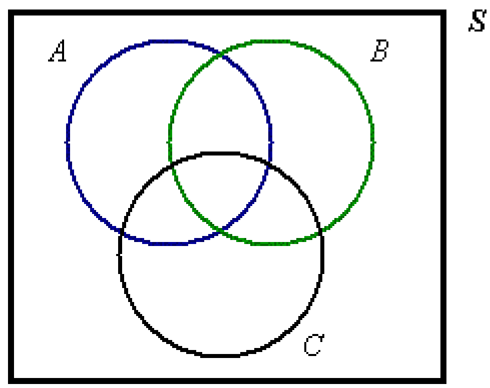
3. At least two of events A, B and C occur.

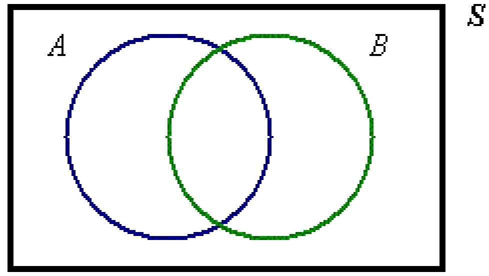


4. Neither B nor C occur.

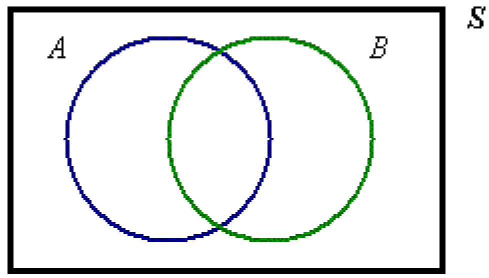


Example 2.07.4 above is an example of **<u>DeMorgan's Laws</u>**:

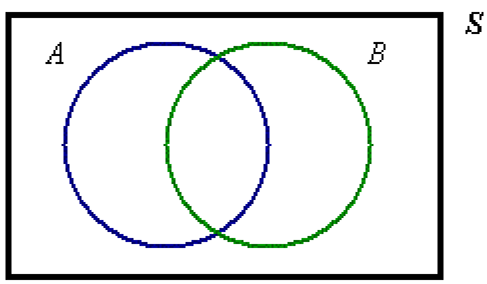
$\sim(A \cup B) =$



$\sim(A \cap B) =$

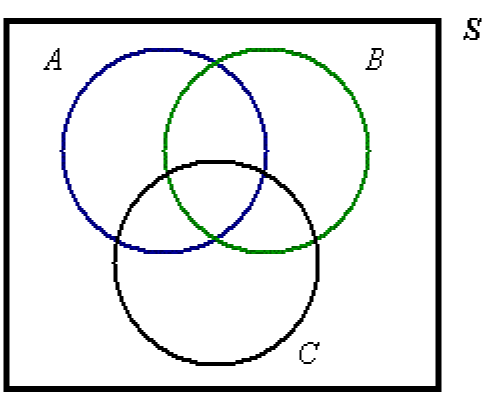


<u>General Addition Law of Probability</u>



$$\text{P}[A \vee B] = \text{P}[A] + \text{P}[B] - \text{P}[A \wedge B]$$

Extended to three events, this law becomes



$$\begin{aligned}\text{P}[A \vee B \vee C] = {} & \text{P}[A] + \text{P}[B] + \text{P}[C] \\ & - \text{P}[A \wedge B] - \text{P}[B \wedge C] - \text{P}[C \wedge A] \\ & + \text{P}[A \wedge B \wedge C]\end{aligned}$$

If two events A and B are **mutually exclusive**
(= **incompatible** = have no common sample points), then
$A \cap B = \emptyset \quad \Rightarrow \quad \text{P}[A \wedge B] = 0$ and the addition law simplifies to

$$\text{P}[A \vee B] = \text{P}[A] + \text{P}[B] .$$

Only when A and B are mutually exclusive may one say " $A \vee B$ " = "$A + B$ ".

<u>Total Probability Law</u>

The total probability of an event A can be partitioned into two mutually exclusive subsets: the part of A that is inside another event B and the part that is outside B :

$$\boxed{\text{P}[A] = \text{P}[A \wedge B] + \text{P}[A \wedge \sim B]}$$

Special case, when $A = S$ and $B = E$:

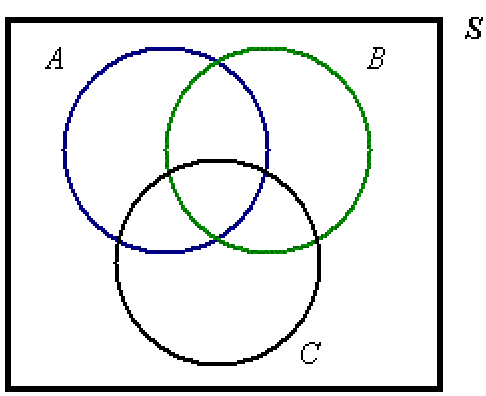
$$\text{P}[S] = \text{P}[S \wedge E] + \text{P}[S \wedge \sim E]$$

$$\Rightarrow \quad 1 = \text{P}[E] + \text{P}[\sim E]$$

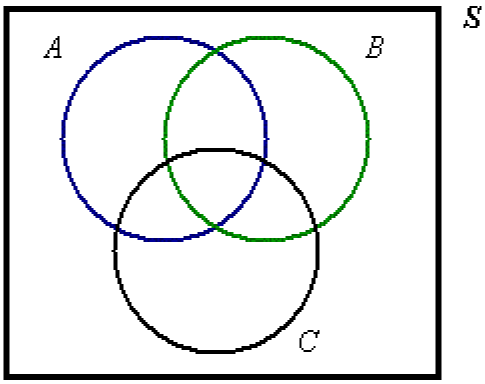
<u>Example 2.08</u>

Given the information that P[ABC] = 2%, P[AB] = 7%, P[AC] = 5% and P[A] = 26%, find the probability that, (of events A,B,C), *only* event A occurs.

A only =



[Example 2.08 continued]



Example 2.09: Roll two fair six-sided dice. The sample space consists of 36 points, as shown below.

Let $\mathbf{E}_1$ = "sum = 7" then

$$P[\mathbf{E}_1] = n(\mathbf{E}_1) \times P[\text{each sample point}] = n(\mathbf{E}_1) \; / \; n(\boldsymbol{S})$$

Let $\mathbf{E}_2$ = "sum > 10" then

$$P[\mathbf{E}_2] =$$

	1	2	3	4	5	6
1						
2						
3						
4						
5						
6						

$\mathbf{E}_1$ and $\mathbf{E}_2$ have no common sample points (disjoint sets; mutually exclusive events) $\Rightarrow$

$$P[\mathbf{E}_1 \textbf{ OR } \mathbf{E}_2] = P[\mathbf{E}_1] + P[\mathbf{E}_2] =$$

The **odds** of $[\mathbf{E}_1 \textbf{ OR } \mathbf{E}_2]$ are:

<u>Example 2.09</u> (continued)

Let $\mathbf{E}_3$ = "at least one '6' " then

P[$\mathbf{E}_3$] =

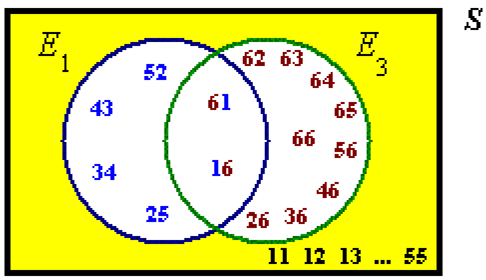
	1	**2**	**3**	**4**	**5**	**6**
1						
2						
3						
4						
5						
6						

P[$\mathbf{E}_1$ **OR** $\mathbf{E}_3$] =

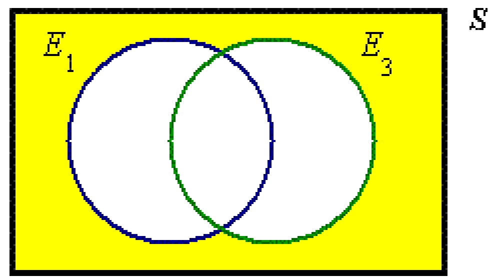
($\because$ common points counted *twice*)

= P[$\mathbf{E}_1$] + P[$\mathbf{E}_3$] – P[$\mathbf{E}_1$ **AND** $\mathbf{E}_3$]

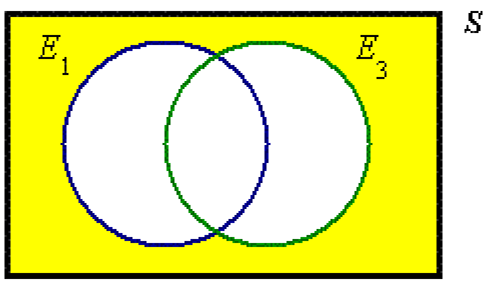
Venn diagram
(each sample point shown):



Venn diagram
(# sample points shown):



Venn diagram for probability:



Some general properties of set/event unions and intersections are listed here:

Commutative: $A \cup B = B \cup A$, $A \cap B = B \cap A$

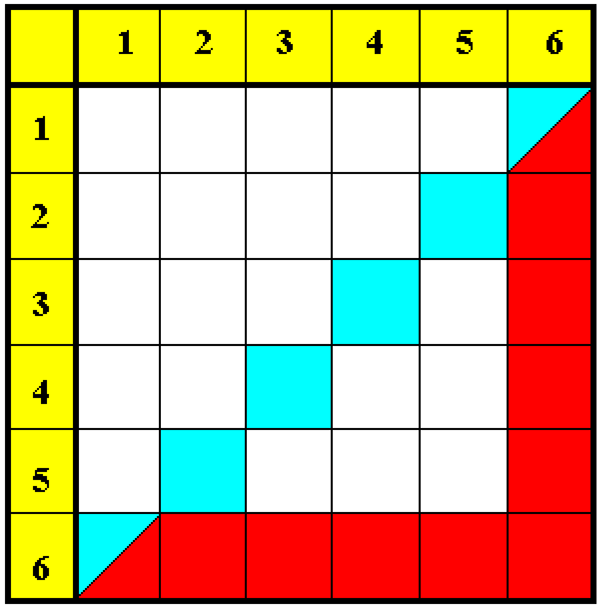
Associative: $(A \cup B) \cup C = A \cup (B \cup C) = A \cup B \cup C$
$(A \cap B) \cap C = A \cap (B \cap C) = A \cap B \cap C$

Distributive: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

In each case, these identities are true for all sets (or events) A, B, C.

<u>Example 2.09</u> (continued)
Find the probability of a total of 7 without rolling any sixes.

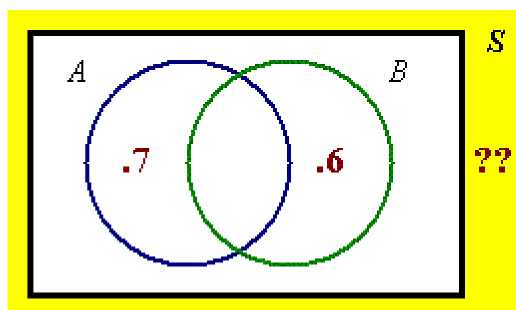
$P[\mathbf{E}_1 \cap \sim\mathbf{E}_3] = P[\mathbf{E}_1] - P[\mathbf{E}_1 \cap \mathbf{E}_3]$ (total probability law)



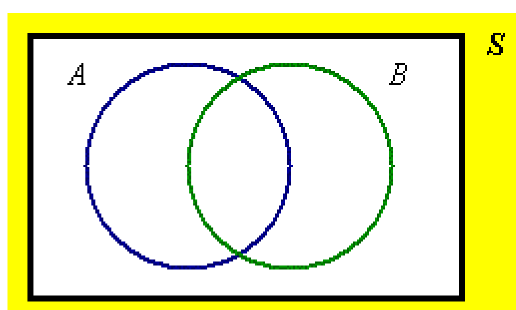
<u>Example 2.10</u>:

Given the information that $P[A \vee B] = .9$, $P[A] = .7$, $P[B] = .6$,
find P[**exactly one** of A, B occurs]

<u>Incorrect</u> labelling of the Venn diagram:



Correct version:



[End of Chapter 2]