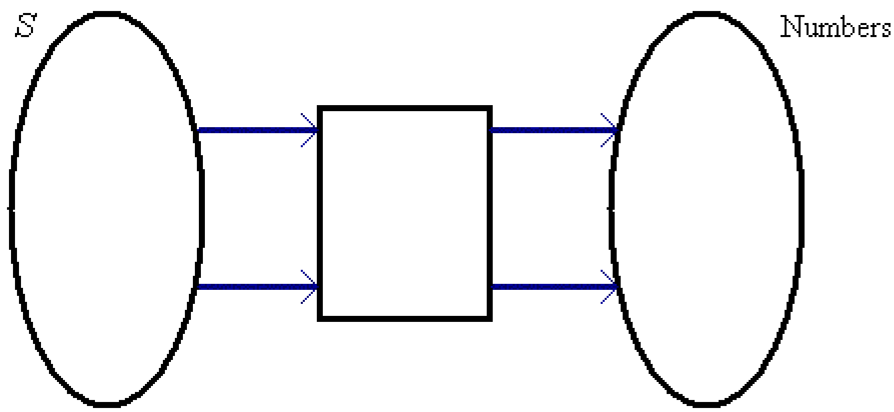
A **random quantity** [r.q.] maps an outcome to a number. [Navidi Section 2.4]
[Devore Sections 3.1-3.3]

Example 5.01:

$P$ = "A student passes ENGI 4421"
$F$ = "That student fails ENGI 4421"

The sample space is $S = \{$




Define $X(P) = 1$ , $X(F) = 0$ , then
$X$ is a random quantity.

Definition: A **Bernoulli random quantity** has only two possible values: 0 and 1.

Example 5.02

Let $Y$ = the sum of the scores on two fair six-sided dice.

$Y(i, j) =$

The possible values of $Y$ are:

Example 5.03

Let $N$ = the number of components tested when one fails.

The possible values of $N$ are:

A set $D$ is **discrete** if

A set $C$ is **continuous** if

---

Examples:

5.03. Set $\mathbb{N}$ = (the set of all natural numbers) is

5.04. $A$ = { $x : 1 \le x \le 2$ and $x$ is real } is

---

A random quantity is **discrete** if its set of possible values is a discrete set.

Each value of a random quantity has some probability of occurring. The set of probabilities for all values of the random quantity defines a function $p(x)$, known as the

**Probability Mass Function**
(or probability function)
(*p.m.f.*):

$$p(x) = \mathrm{P}[X = x]$$

Note: $X$ is a random quantity, but $x$ is a particular value of that random quantity.

All probability mass functions satisfy both of these conditions:

[ ] and [ ]

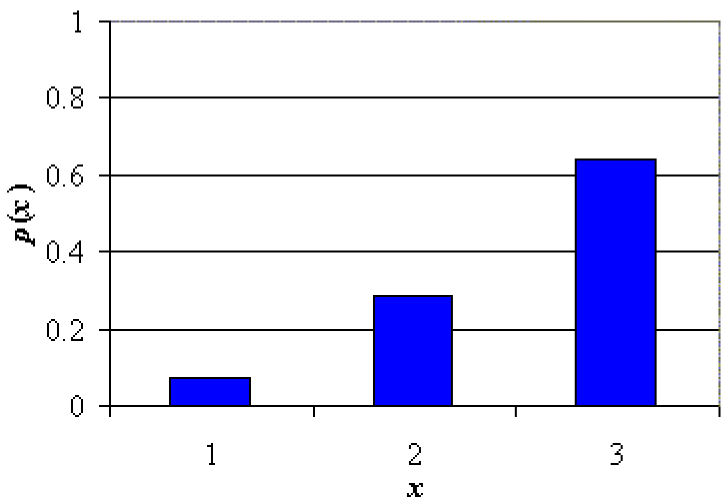
Example 5.05

$$p(x) = \begin{cases} c x^2 & x = 1, 2, 3 \\ 0 & \text{otherwise} \end{cases} \quad \leftarrow \text{ [NOTE: may omit this branch]}$$

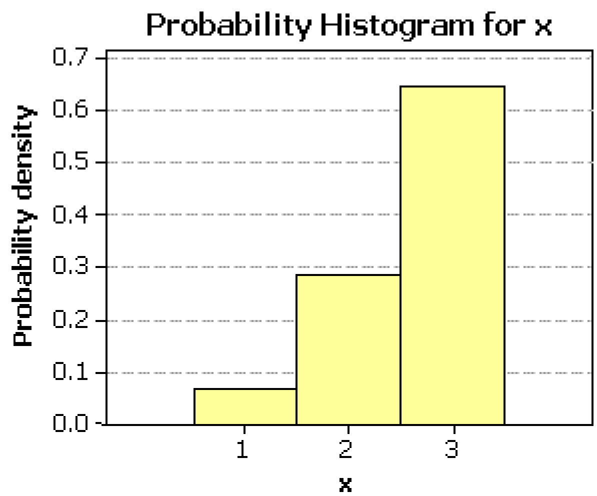
[ $p(x) = 0$ is assumed for all $x$ not mentioned in the definition of $p(x)$.]

$p(x)$ is a probability mass function. Find the value of the constant $c$ .

Bar Chart:




or



Example 5.06

Find the p.m.f. for $X$ = (the number of heads when two fair coins are tossed).

Let $H_i$ = head on coin $i$ and $T_i$ = tail on coin $i$.

The possible values of $X$ are $X$ =

## The Discrete Uniform Probability Distribution

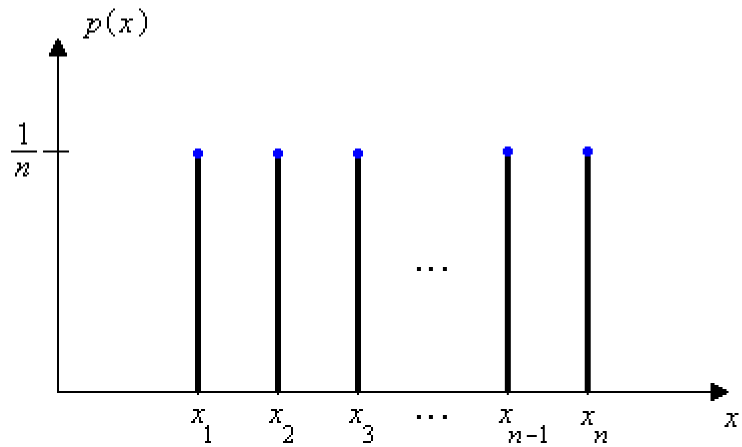
A random quantity $X$, whose $n$ possible values $\{x_1, x_2, x_3, \ldots, x_n\}$ are all equally likely, possesses a discrete uniform probability distribution.

$$P[X = x_i] = \frac{1}{n} \qquad (i = 1, 2, \ldots, n)$$

An example is $X$ = (the score on a fair standard six-sided die),

for which $n = 6$ and $x_i = i$ .

Line graph:



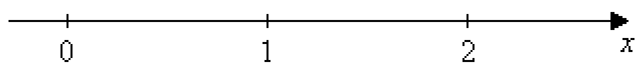
**Cumulative Distribution Function** (c.d.f.)

$$F(x) = \mathrm{P}[X \le x] = \sum_{y:y \le x} p(y)$$

Example 5.07

Find the cumulative distribution function for
$X$ = (the number of heads when two fair coins are tossed).

The possible values of $X$ are 0, 1 and 2.




$F(0) = \mathrm{P}[X \le 0] = p(0) = \frac{1}{4}$

$F(1) = \mathrm{P}[X \le 1] = \mathrm{P}[X < 1] + \mathrm{P}[X = 1]$

$= F(0) + p(1)$

$=$

$F(2) = \mathrm{P}[X \le 2]$

$= F(1) + p(2)$

$=$

When $x < 0$, $F(x) = \mathrm{P}[X \le x] \le \mathrm{P}[X < 0] = 0 \quad \Rightarrow \quad F(x) = 0$

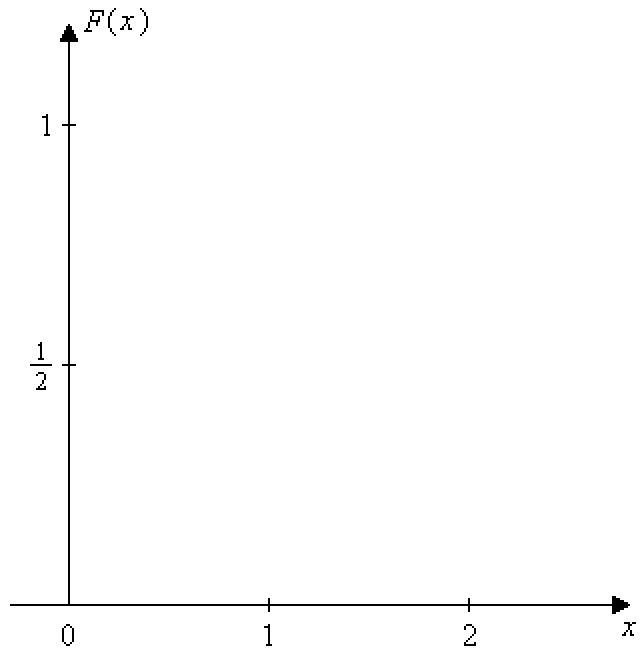
When $x > 2$, $F(x) = \mathrm{P}[X \le x] = F(2) + \mathrm{P}[2 < X \le x] =$

When $1 < x < 2$, $F(x) = \mathrm{P}[X \le x] = F(1) + \mathrm{P}[1 < X \le x] =$

Thus

$$F(x) = \begin{cases} 0 & \text{if} \quad x < 0 \\ 1/4 & \text{if} \quad 0 \le x < 1 \\ 3/4 & \text{if} \quad 1 \le x < 2 \\ 1 & \text{if} \quad 2 \le x \end{cases}$$

The graph of the c.d.f. is:




**In general**, the graph of a discrete c.d.f. :

- is always non-decreasing
- is level between consecutive possible values (staircase appearance)
- has a finite discontinuity at each possible value (step height = $p(x)$ )
- rises in steps from $F(x) = 0$ to $F(x) = 1$.

<u>Example 5.08</u> (the inverse of the preceding problem):

Find the probability mass function $p(x)$ given the cumulative distribution function

$$F(x) = \begin{cases} 0 & \text{if} \quad x < 0 \\ 1/4 & \text{if} \quad 0 \le x < 1 \\ 3/4 & \text{if} \quad 1 \le x < 2 \\ 1 & \text{if} \quad 2 \le x \end{cases}$$

Steps (= possible values) are at $x = 0, 1, 2$ only.

---

In general,

$$\boxed{P[a < X \le b] = F(b) - F(a)}$$

If $a$ , $b$ and **all** possible values are integers, then

$$\boxed{P[a \le X \le b] = F(b) - F(a-1)} \quad \text{and} \quad \boxed{p(a) = P[X = a] = F(a) - F(a-1)}$$

---

Example 5.09

Find and sketch the *c.d.f.* for $X$ = (the score upon rolling a fair standard die once).

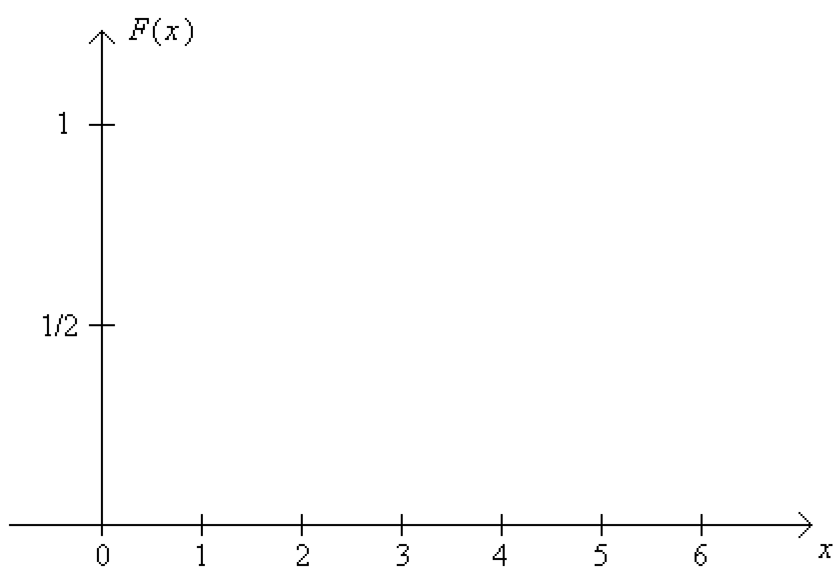
The *p.m.f.* is a uniform distribution

$$p(x) =$$

Thus $F(x)$ increases from 0 to 1/6 at $x = 1$ and increases by steps of 1/6 at each subsequent integer value until $x = 6$. It follows easily that

$$F(x) =$$

The graph of $F(x)$ has the classic staircase appearance of the cumulative distribution function of a discrete random quantity.



**Expected value of a random quantity**

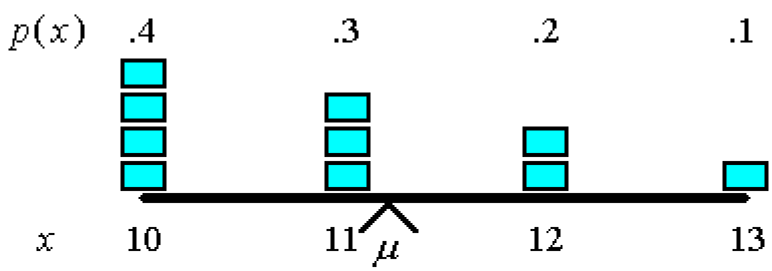
Example 5.10:

The random quantity $X$ is known to have the p.m.f.

| $x$ | 10 | 11 | 12 | 13 |
|---|---|---|---|---|
| $p(x)$ | .4 | .3 | .2 | .1 |

If we measure values for $X$ many times, what value do we expect to see on average?

Treat the values of $p(x)$ as point masses of probability:




The **expected value** $\mathrm{E}[X]$ (= **population mean** $\mu$) is at the fulcrum (balance point) of the beam.

Taking moments about $x = 10$:

In general, for any random quantity $X$ with a discrete probability mass function $p(x)$ and a set of possible values $D$, the population mean $\mu$ of $X$ (and the expected value of $X$) is

$$\mathrm{E}[X] = \mu_X = \sum_{x \in D} x \cdot p(x)$$

**Shortcut:** If $X$ is symmetric about $x = a$, then $\mathrm{E}[X] =$

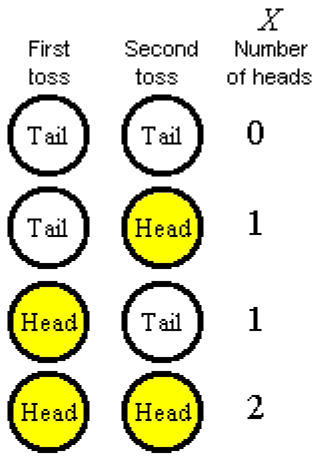
Example 5.11:

Let $X$ = the number of heads when a coin has been tossed twice. Find E[$X$].

Solution:

List the all the possible combinations.

$\rightarrow$ the probability mass function of the distribution of $X$.

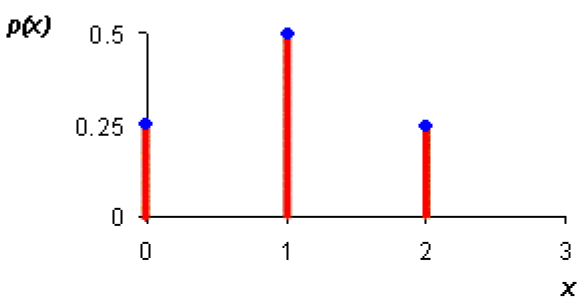



$$p(x) = \begin{cases} & (x=0) \\ & (x=1) \\ & (x=2) \end{cases}$$

E[$X$] =

Alternative solution:

Graph of $p(x)$:




$p(x)$ is symmetric about $x$ = .

Therefore, E[$X$] =

**The expected value of a function**

**Definition:**

If the random quantity $X$ has set of possible values $D$ and p.m.f. $p(x)$, then the expected value of any function $h(X)$, denoted by $\mathrm{E}\left[h(X)\right]$, is computed by

$$\boxed{\mathrm{E}\left[h(X)\right] = \sum_{\text{all } x} h(x) \cdot p(x)}$$

$\mathrm{E}\left[h(X)\right]$ is computed in the same way that $\mathrm{E}\left[X\right]$ itself is, except that $h(x)$ is substituted in place of $x$.

**Special case:**

$h(x) = ax + b \quad \Rightarrow$ $$\boxed{\mathrm{E}\left[aX + b\right] = a\,\mathrm{E}\left[X\right] + b}$$

Proof:

Example 5.12:
$C$ = tomorrow's temperature high in °C
$F$ = tomorrow's temperature high in °F
Given $E[C] = 10$, find $E[F]$.

---

## The variance of $X$

The quantity usually employed to measure the spread in the values of a random quantity $X$ is the **population variance** $\mathrm{V}\left[X\right] = \sigma^2 = \frac{1}{N}\sum_x (x-\mu)^2$

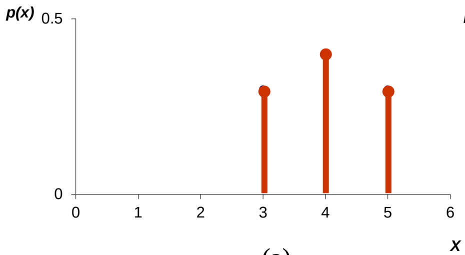
Let $X$ have probability mass function $p(x)$ and expected value $\mu$. Then

$$\boxed{\mathrm{V}\left[X\right] = \sum_x (x-\mu)^2\, p(x) = \mathrm{E}\left[(X-\mu)^2\right]}$$

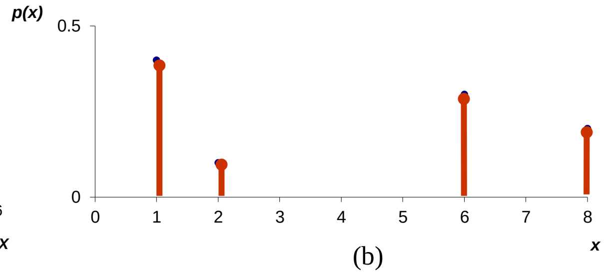
The **standard deviation** of $X$ is $\sigma = \sqrt{\mathrm{V}[X]}$

<u>Example 5.13</u>:

Two different probability distributions [below] share the same mean $\mu = 4$




(a)




(b)

If $X$ has the p.m.f. (as shown in Figure (a))

| $x$ | 3 | 4 | 5 |
|---|---|---|---|
| $p(x)$ | .3 | .4 | .3 |

$\mu =$

V[$X$] =

If $X$ has the p.m.f (as shown in Figure (b))

| $x$ | 1 | 2 | 6 | 8 |
|---|---|---|---|---|
| $p(x)$ | .4 | .1 | .3 | .2 |

$\mu =$

V[$X$] =

Example 5.14:

Let $X$ = number of heads when a coin has been tossed twice. Find V[$X$].

$$\mathrm{V}[X] = \mathrm{E}\left[(X-\mu)^2\right] =$$

---

**A shortcut formula for variance**

$$\boxed{\mathrm{V}[X] = \mathrm{E}\left[X^2\right] - \left(\mathrm{E}[X]\right)^2}$$

Proof:

**Note:** $\mathrm{E}\left[f(X)\right] \neq f\left(\mathrm{E}[X]\right)$ unless $f(x)$ is linear and/or $X$ is constant.

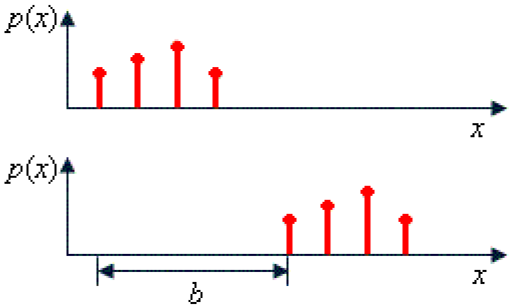
Example 5.14 (continued):
Let $X$ = number of heads when a coin has been tossed twice.
Find V[$X$] using the shortcut formula.

The shortcut is more convenient when $\mu$ is not an integer.

**Rules of variance** [part of Navidi Section 2.5]

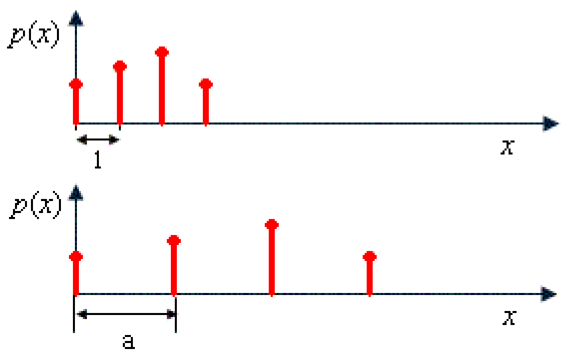
Example 5.15:

Do the distributions in the following two figures have the same variance or not?




Example 5.16:

Do the distributions in the following two figures have the same variance or not?




$$\boxed{\mathrm{V}[aX+b] = a^2\mathrm{V}[X]}$$

Proof:

The addition of the constant *b* does not affect the variance, because the addition of *b* changes the location (and therefore mean value) but not the spread of values.

[Space for any additional notes]