**Central Limit Theorem** [Navidi, section 4.11; Devore sections 5.3-5.4]

If $X_i$ is **not** normally distributed, but $\text{E}[X_i] = \mu$, $\text{V}[X_i] = \sigma^2$ and $n$ is large (approximately 30 or more), then, to a good approximation,

$$\bar{X} \sim \text{N}\left(\mu, \frac{\sigma^2}{n}\right)$$

At "http://www.engr.mun.ca/~ggeorge/4421/demos/CLT.html" is a demonstration web program to illustrate how the sample mean approaches a normal distribution even for highly non-normal discrete distributions of $X$.

Consider the exponential distribution, whose p.d.f. (probability density function) is

$$f(x; \lambda) = \lambda e^{-\lambda x}, \quad (x \ge 0, \lambda > 0) \quad \Rightarrow \quad \text{E}[X] = \frac{1}{\lambda}, \quad \text{V}[X] = \frac{1}{\lambda^2}$$

It can be shown that the exact p.d.f. of the sample mean for sample size $n$ is

$$f_{\bar{X}}(x; \lambda, n) = \frac{\lambda n (\lambda n x)^{n-1} e^{-\lambda n x}}{(n-1)!}, \quad (x \ge 0, \lambda > 0, n \in \mathbb{N})$$

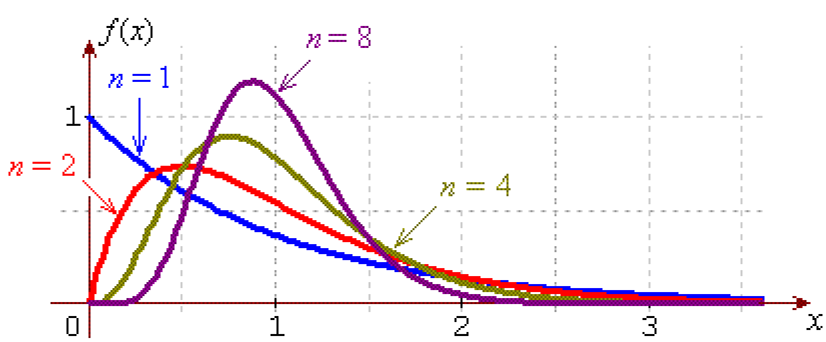
with mean and variance $\text{E}[\bar{X}] = \dfrac{1}{\lambda}$, $\text{V}[\bar{X}] = \dfrac{1}{n\lambda^2}$.

[A non-examinable derivation of this p.d.f. is available at "http://www.engr.mun.ca/~ggeorge/4421/demos/cltexp2.pdf".]

For illustration, setting $\lambda = 1$, the p.d.f. for the sample mean for sample sizes $n$ = 1, 2, 4 and 8 are:

$n = 1$: $\quad f(x) = e^{-x}$ $\qquad\qquad n = 2$: $\quad f_{\bar{X}}(x) = 4xe^{-2x}$

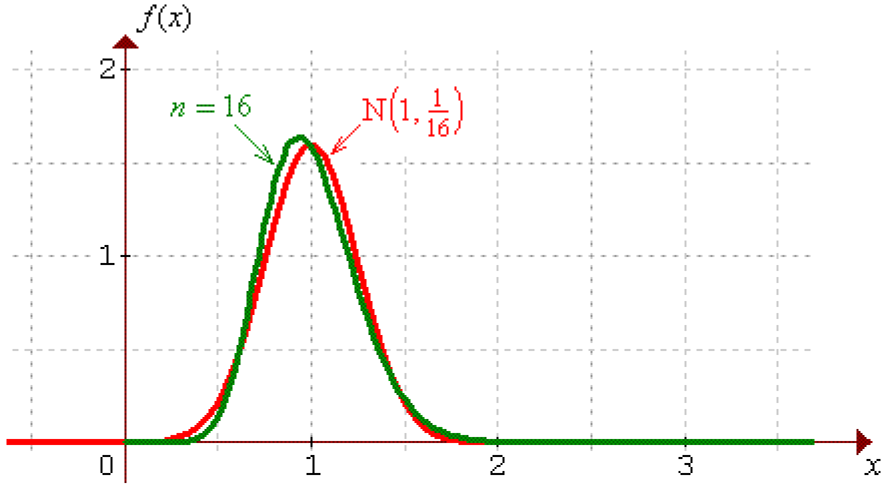
$n = 4$: $\quad f_{\bar{X}}(x) = \dfrac{4(4x)^3 e^{-4x}}{3!}$ $\qquad\qquad n = 8$: $\quad f_{\bar{X}}(x) = \dfrac{8(8x)^7 e^{-8x}}{7!}$




The population mean $\mu = \text{E}[X] = 1$ for all sample sizes.
The variance and the positive skew both diminish with increasing sample size.
The mode and the median approach the mean from the left.

For a sample size of $n = 16$, the sample mean $\bar{X}$ has the p.d.f.

$$f_{\bar{X}}(x) = \frac{16(16x)^{15} e^{-16x}}{15!} \text{ and parameters } \mu = \mathrm{E}\left[\bar{X}\right] = 1 \text{ and } \sigma^2 = \mathrm{V}\left[\bar{X}\right] = \frac{1}{16}.$$




A plot of the exact p.d.f is drawn here, together with the normal distribution that has the same mean and variance. The approach to normality is clear. Beyond $n = 40$ or so, the difference between the exact p.d.f. and the Normal approximation is negligible.

It is generally the case that, whatever the probability distribution of a random quantity may be, the probability distribution of the sample mean $\bar{X}$ approaches normality as the sample size $n$ increases. For most probability distributions of practical interest, the normal approximation becomes very good beyond a sample size of $n \approx 30$.

---

Example 11.01

A random sample of 100 items is drawn from an exponential distribution with parameter $\lambda = 0.04$. Find the probabilities that

(a) a single item has a value of more than 30;

(b) the sample mean has a value of more than 30.

---

(a)

Example 11.01 (continued)

(b)

**Sample Proportions**

A Bernoulli random quantity has two possible outcomes:

$x = 0$ (= "failure") with probability $q = 1 - p$

and $x = 1$ (= "success") with probability $p$ .

Suppose that all elements of the set $\{X_1, X_2, X_3, \ldots, X_n\}$ are independent Bernoulli random quantities, (so that the set forms a **random sample**).

Let $T = X_1 + X_2 + X_3 + \ldots + X_n =$ number of successes in the random sample

and $\hat{P} = \dfrac{T}{n}$ = proportion of successes in the random sample,

then $T$ is

<u>Example 11.02</u>

55% of all customers prefer brand *A*.
Find the probability that a majority in a random sample of 100 customers does **not** prefer brand *A*.

**<u>Confidence Intervals (One Sample)</u>** [Navidi, sections 5.1-5.3; Devore chapter 7]

So far we have constructed probability statements on how likely certain sample values are, given knowledge of the population from which the random sample came. Now we shall reverse that situation: we have a known sample in front of us, from which we can infer the values of the parameters of the population from which the sample was drawn. This is the realm of **inferential statistics**.

If the random quantity $X$ is such that $X \sim \mathrm{N}\left(\mu, \sigma^2\right)$, then it is highly unlikely that $X$ will be more than three standard deviations away from its mean:

$$\mathrm{P}\left[\,|X-\mu|>3\sigma\right] = 2\,\mathrm{P}\left[X<\mu-3\sigma\right] = 2\,\mathrm{P}\left[Z<-3\right] = 2\,\Phi\left(-3.00\right) < .003$$

More than 99.7% of the time, $X$ will be closer than three standard deviations to its mean.

For a sufficiently large random sample, the central limit theorem assures us that the sample mean $\overline{X} \sim \mathrm{N}\left(\mu, \frac{\sigma^2}{n}\right)$ (either exactly or to an excellent approximation).

Therefore we have 99.7% confidence that an observed sample mean $\bar{x}$ is within three standard errors $\frac{\sigma}{\sqrt{n}}$ of the population mean $\mu$. This line of reasoning allows us to replace a point estimate by a range of plausible values of an unknown parameter – a **confidence interval**.

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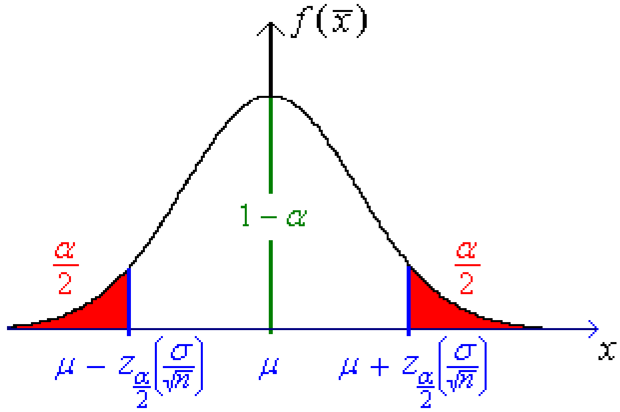
More generally, when $\overline{X} \sim \mathrm{N}\left(\mu, \frac{\sigma^2}{n}\right)$,




$$\mathrm{P}\left[\mu - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} < \overline{X} < \mu + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right] = 1-\alpha$$

---

The **confidence interval estimator** for $\mu$ (at a level of confidence of $(1-\alpha)$) is

$$\bar{X} - z_{\alpha/2}\left(\frac{\sigma}{\sqrt{n}}\right) < \mu < \bar{X} + z_{\alpha/2}\left(\frac{\sigma}{\sqrt{n}}\right)$$

The $(1-\alpha)$ confidence interval estimator for $\mu$ is a random interval

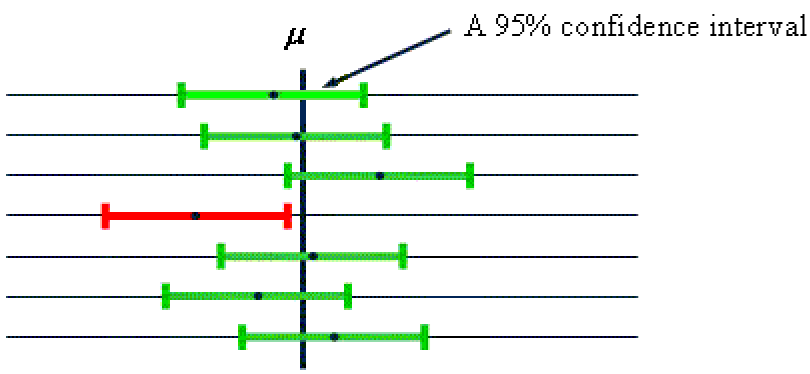
$$\left[\bar{X} - z_{\alpha/2}\left(\frac{\sigma}{\sqrt{n}}\right), \ \bar{X} + z_{\alpha/2}\left(\frac{\sigma}{\sqrt{n}}\right)\right]$$

The probability is $(1 - \alpha)$ that the above random interval includes the true value of $\mu$. . $(1-\alpha)$ of all random samples will produce an inequality, (the $(1-\alpha)$ **confidence interval estimate** for $\mu$)

$$\bar{x} - z_{\alpha/2}\left(\frac{\sigma}{\sqrt{n}}\right) < \mu < \bar{x} + z_{\alpha/2}\left(\frac{\sigma}{\sqrt{n}}\right)$$

that is true. Note that the confidence interval estimate contains no random quantities at all! The statement is either absolutely certain to be true or absolutely certain to be false, (depending on the values of $\mu, \sigma, \bar{x}, n$ and $\alpha$).

**Interpretation of a confidence interval** [ = confidence interval **estimate** ]




Only 5% of all 95% confidence interval estimates for $\mu$ fail to include $\mu$.

A concise expression for the C.I. (confidence interval estimate for $\mu$) is

$$\bar{x} \pm z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$$

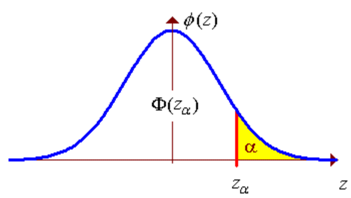
**A Bayesian view of interval estimation:**

If the only quantity among $\{\mu, \sigma, \bar{x}, n$ and $\alpha\}$ that we don't know is $\mu$, then represent the unknown $\mu$ by the random quantity $A$. Then

---

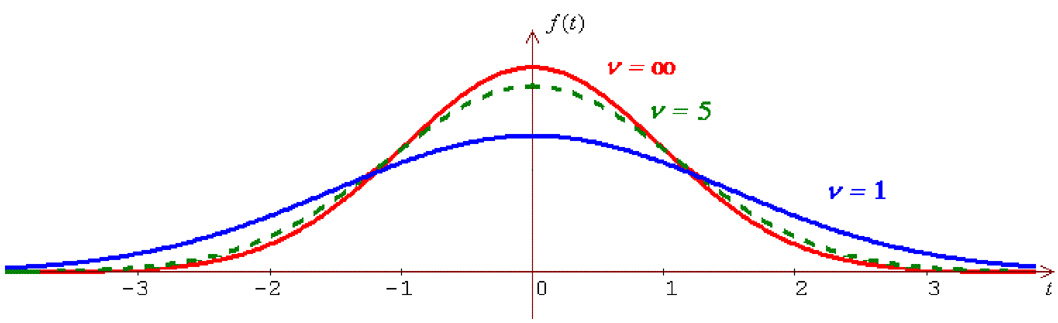
**A note about the standard normal distribution and the *t* distribution**

Let $Z \sim N(0, 1)$ (standard normal distribution), so that $P[Z \le z] = \Phi(z)$ (cumulative distribution function for the standard normal distribution).
Then the $(1-\alpha)\times 100^{\text{th}}$ percentile of the standard normal distribution is $z_\alpha$, which satisfies $P[Z > z_\alpha] = \alpha$.
It also follows that $1-\Phi(z_\alpha) = \Phi(-z_\alpha) = \alpha$.




$\Phi(z_\alpha) = 1-\alpha$

The *t* distribution with $\nu$ degrees of freedom is also a bell shaped curve, with a mean, median and mode at $t = 0$, but with a greater variance than the standard normal distribution. As the number of degrees of freedom increases, the *t* distribution approaches the *z* (standard normal) distribution. The graphs of $t_1$ and $t_5$ are shown here, together with $\phi(z)$, which is indistinguishable to the eye from $t_\nu$ for $\nu$ above 30 or so.




Therefore $\lim_{\nu\to\infty} t_{\alpha,\nu} = t_{\alpha,\infty} = z_\alpha$.

Use the $t$ distribution only if the true population variance $\sigma^2$ is unknown.

---

To find the $(1-\alpha)\times100^{\text{th}}$ percentile $z_\alpha$, use the final row in the table of critical values of the $t$ distribution (on page 17-02 or the inside back cover of the textbook):
$z_\alpha = t_{\alpha,\infty}$.

The final row of the $t$ tables is

| ν \ α | **0.1** | **0.05** | **0.025** | **0.01** | **0.005** |
|---|---|---|---|---|---|
| ∞ | **1.28155** | **1.64485** | **1.95996** | **2.32635** | **2.57583** |

Therefore $P[Z > 1.645] \approx .05$ or equivalently $z_{.05} \approx 1.645$;
$P[Z > 1.960] \approx .025$ or equivalently $z_{.025} \approx 1.960$; etc.

---

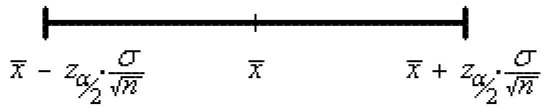
Example 11.03

The rate of energy loss $X$ (watt) in a motor is known to be a normally distributed random quantity with standard deviation $\sigma = 3.0\text{ W}$. A random sample of 100 such motors produces a sample mean rate of energy loss of 58.3 W. Find a 99% confidence interval estimate for the true mean rate of energy loss $\mu$.

---

Example 11.03 (continued)

How large must $n$ be for the width of the 99% confidence interval estimate for $\mu$ to be less than 1.0?




**Choice of sample size**

The width of the confidence interval $\left( \bar{x} - z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}},\ \bar{x} + z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}} \right)$ is

$$w = 2 \cdot z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}} \qquad \Rightarrow \qquad \boxed{n = \left( 2 z_{\alpha/2} \cdot \frac{\sigma}{w} \right)^2}$$

The sample size is inversely related to the square of the desired width.

---

**<u>Endpoints of a (1 – α) CI for $\mu$</u>:**

(a) $\sigma^2$ known:

(b) $\sigma^2$ unknown, $n$ large:

(c) $\sigma^2$ unknown, $n$ small:

When $n$ is small, $X$ must be nearly (or exactly) normal.

---

Example 11.04

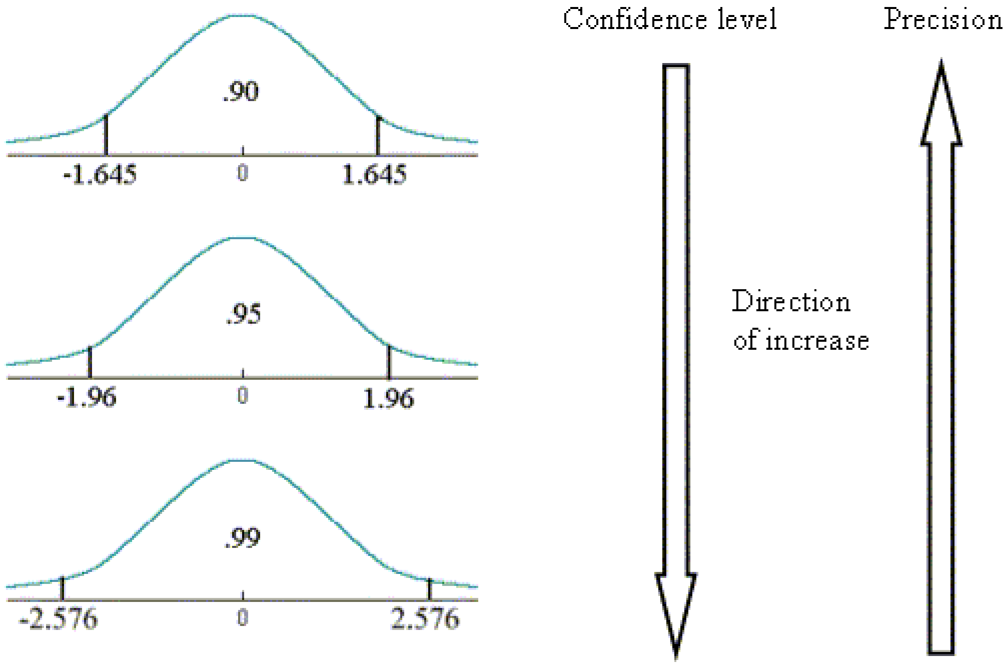
The lifetime $X$ of a particular brand of filaments is known to be normally distributed. A random sample of six filaments is tested to destruction and they are found to last for an average of 1,008 hours with a sample standard deviation of 6.2 hours.

(a) Find a 95% confidence interval estimate for the population mean lifetime $\mu$.

(b) Is the evidence consistent with $\mu \neq 1000$ ?

(c) Is the evidence consistent with $\mu > 1000$ ?

**Properties of a confidence interval**




If we think of the width of the confidence interval as specifying its precision, then the confidence level (or reliability) of the interval is inversely related to its precision.

---

**Estimation of Population Proportion**

When a random sample of size $n$ is drawn from a population in which a proportion $p$ of the items are "successes", then, as we saw on page 11.04,

$$\hat{P} \sim N\left(p, \frac{pq}{n}\right)$$

for sufficiently large $np$ and $nq$, (namely, $np > 10$ and $nq > 10$ ).

Compare this with $\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$, for which the corresponding confidence interval has endpoints $\bar{x} \pm z_{\alpha/2} \cdot \left(\frac{s}{\sqrt{n}}\right)$.

However, the variance $\frac{pq}{n}$ is unknown because $p$ and $q$ are unknown.

An obvious remedy is to replace the unknown parameters $p$ and $q$ by their point estimates $\hat{p}$ and $\hat{q}$.

---

Therefore, a simple 100(1 – α)% confidence interval estimator for $p$ is

$$P \pm z_{\alpha/2}\sqrt{\frac{PQ}{n}}$$

and the 100(1 – α)% confidence interval **estimate** for $p$ is

$$\hat{p} \pm z_{\alpha/2}\sqrt{\frac{\hat{p}\hat{q}}{n}}$$

However, these confidence intervals can exhibit significant errors when either $np$ or $nq$ is much less than 100. During the 1990's, more reliable confidence intervals for $p$ were developed. One of them is (Devore, sixth edition, section 7.2, page 266) :

$$\frac{\hat{p} + \dfrac{z_{\alpha/2}^{2}}{2n} \pm z_{\alpha/2}\sqrt{\dfrac{\hat{p}\hat{q}}{n} + \dfrac{z_{\alpha/2}^{2}}{4n^{2}}}}{1 + \dfrac{z_{\alpha/2}^{2}}{n}}$$

Another interval, from the Navidi textbook (page 339), is the Agresti-Coull interval.
If $x$ is the observed number of successes in a random sample of $n$ independent Bernoulli trials, then define $x^* = x + 2$ and $n^* = n + 4$ so that

$$p^* = \frac{x^*}{n^*} = \frac{x+2}{n+4} \quad \text{and} \quad q^* = 1 - p^*$$

Then the 100(1 – α)% confidence interval **estimate** for $p$ is

$$p^* \pm z_{\alpha/2}\sqrt{\frac{p^* q^*}{n^*}}$$

It turns out that the Bayesian point estimate for $p$ is $p^*$, not $\hat{p}$ .

Example 11.05
From a random sample of one thousand silicon wafers, 750 pass a quality control test. Find a 99% confidence interval estimate for $p$ (the true proportion of wafers in the population that are good).

---

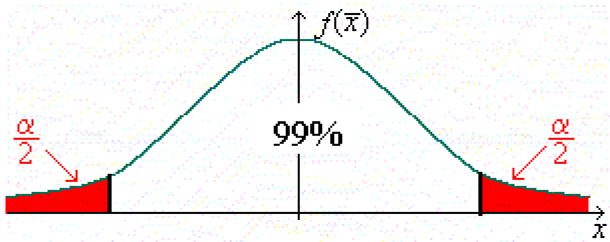
$n = 1000$ and $x = 750$

$$\Rightarrow \quad \hat{p} = \frac{x}{n} = \frac{750}{1000} = \frac{3}{4}$$

$$\Rightarrow \quad \hat{q} = 1 - \hat{p} = \frac{1}{4}$$

$$\frac{\alpha}{2} = .005$$

$$z_{.005} = t_{.005,\infty} \approx 2.576$$




Endpoints of the C.I.:

$$\hat{p} \pm z_{\alpha/2}\sqrt{\frac{\hat{p}\,\hat{q}}{n}} = .75 \pm 2.576\sqrt{\frac{.75\times.25}{1000}} = .75 \pm .035\,27...$$

Therefore the 99% confidence interval estimate for $p$ is

**$71.5\% \le p \le 78.5\%$**

correct to three significant figures.

---

Using the more precise Agresti-Coull version of the confidence interval yields

$$x^* = 750 + 2 = 752, \quad n^* = 1000 + 4 = 1004 \quad \Rightarrow \quad p^* = \frac{x^*}{n^*} = \frac{752}{1004} = .749\,003\ldots$$

$$\Rightarrow \quad \sqrt{\frac{p^* q^*}{n^*}} = \sqrt{\frac{.749\ldots\times.250\ldots}{1004}} = .013\,683\ldots$$

The 99% CI is therefore

$$p^* \pm z_{.005}\sqrt{\frac{p^* q^*}{n^*}} = .749\ldots \pm 2.576\times.0136\ldots = .749\,003\ldots \pm .035\,247\ldots$$

$\Rightarrow$ **$71.4\% \le p \le 78.4\%$**

## (1−α)×100% Bayesian Confidence Interval for $\mu$

[not in the Navidi or Devore textbooks]

Suppose that previous evidence leads us to believe that $\mu = \mu_o$. The strength of this belief is represented by the variance $\sigma_o^2$ (lower variance corresponds to stronger belief). We wish to update that estimate after a random sample of size $n$ has been examined. Assume that $n \gg 30$ (so that the Central Limit Theorem will apply).

**Prior distribution:**

$$\bar{X} \sim \mathrm{N}\left( \mu_o , \sigma_o^{\,2} \right)$$

**New evidence:**

Sample size = $n$
Sample mean = $\bar{x}$
Sample standard deviation = $s$

Calculate

$$\boxed{\mu^* = \frac{w_d\, \bar{x} + w_o\, \mu_o}{w_d + w_o}}, \quad \boxed{\left(\sigma^*\right)^2 = \frac{1}{w_d + w_o}}$$

where $w_d$, $w_o$ are the weights of the data and original information respectively, given by

$$\boxed{w_d = \frac{1}{\left(\frac{s^2}{n}\right)}, \quad w_o = \frac{1}{\sigma_o^{\,2}}}$$

**Posterior distribution:**

$$\bar{X} \sim \mathrm{N}\left( \mu^*, \left(\sigma^*\right)^2 \right)$$

→ $(1-\alpha)\times 100\%$ Bayesian interval for $\mu$:

$$\mu = \mu^* \pm z_{\alpha/2}\, \sigma^*$$

Compare with the classical $(1-\alpha)\times 100\%$ confidence interval for $\mu$:

$$\mu = \bar{x} \pm z_{\alpha/2} \frac{s}{\sqrt{n}} \quad (n >> 30) \quad \text{or} \quad \mu = \bar{x} \pm z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$$

In many applications, the Bayesian interval is often narrower than the classical confidence interval, because the Bayesian interval incorporates more information (previous evidence or belief about the true value of $\mu$).

[Note: it is easy to show that as $\sigma_o^2 \to \infty$ (or if $\bar{x} = \mu_o$ then), $\mu^* = \bar{x}$
and that as $\sigma_o^2 \to \infty$, $\left(\sigma^*\right)^2 \to s^2/n$, which are the classical expressions.]

## Examples of Bayesian Confidence Intervals

These examples are modifications of the previous examples of classical confidence intervals for $\mu$.

Example 11.06 (modification of Example 11.03)

The rate of energy loss $X$ (watt) in a motor is known to be a normally distributed random quantity and prior experience suggests that the mean is $\mu = 60\text{ W}$ with standard deviation $\sigma = 3.0\text{ W}$. A random sample of 100 such motors produces a sample mean rate of energy loss of 58.3 W with sample standard deviation 2.8 W. Find a 99% confidence interval estimate for the true mean rate of energy loss $\mu$.

---

Example 11.07 (modification of Example 11.04)

The lifetime $X$ of a particular brand of filaments is known to be normally distributed. Prior experience suggests that $\mu = 1000$ and $\sigma = 6.0$. A random sample of six filaments is tested to destruction and they are found to last for an average of 1,008 hours with a sample standard deviation of 6.2 hours.

(a) Find a 95% confidence interval estimate for the population mean lifetime $\mu$.

(b) Is the evidence consistent with $\mu \neq 1000$?

---

[End of Chapter 11]

[Space for Additional Notes]