ENGI 4430 Advanced Calculus for Engineering
Faculty of Engineering and Applied Science

Problem Set 2 Questions

[Parametric Vector Functions]

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1. For the curve in $\mathbb{R}^2$ that is given in parametric form by

$$\vec{\mathbf{r}}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} t^2 \\ t^3 \end{bmatrix}$$

(a) Sketch the curve.
(b) Find the unit tangent $\hat{\mathbf{T}}$, the unit normal $\hat{\mathbf{N}}$ and the unit binormal $\hat{\mathbf{B}}$.
(c) Explain why the unit tangent and the unit normal are undefined at the origin. Can the unit binormal be defined at the origin?
(d) Find the exact distance along the curve from the origin to the point (1, 1).

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2. In problem set 1 question 1 we found that the curve whose Cartesian equation is

$$\left(x^2 + y^2\right)^{3/2} = 2x^2$$

has the much simpler form in polar coordinates $r = 1 + \cos 2\theta$.
Find the perimeter of one loop of this curve.
Leave your answer in the form of a definite integral.

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3. For the curve in problem set 1 question 5, whose equation in polar form is

$$r^2 = 4\cos 3\theta$$

(a) Evaluate the area enclosed by any one loop of the curve.
(b) Show that the total arc length along any one loop of the curve can be expressed as

$$L = 2\int_0^{\pi/6} \sqrt{9\sec 3\theta - 5\cos 3\theta}\, d\theta$$

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4. For the curve, whose equation is expressed in terms of the parameter t by

$$\vec{\mathbf{r}}(t) = 3e^{-t}\cos t\,\hat{\mathbf{i}} + 3e^{-t}\sin t\,\hat{\mathbf{j}} + 4e^{-t}\,\hat{\mathbf{k}}$$

(a) Find the distance along the curve from the point (3, 0, 4) to the origin.
(b) Find the curvature $\kappa(t)$ and the radius of curvature $\rho(t)$.
(c) Describe the behaviour of the curve as the parameter $t \to \infty$.

5. There are three formulæ connecting arc length and the unit tangent, unit principal normal and unit binormal vectors, known as the *Frenet-Serret* formulæ.

(a) Show that the first Frenet-Serret formula is $\frac{d\hat{\mathbf{T}}}{ds} = \kappa \hat{\mathbf{N}}$, where s is arc length and κ is the curvature.

(b) Use $\hat{\mathbf{B}} = \hat{\mathbf{T}} \times \hat{\mathbf{N}}$ to show that $\frac{d\hat{\mathbf{B}}}{ds}$ is perpendicular to $\hat{\mathbf{T}}$.

(c) Use $\hat{\mathbf{B}} \bullet \hat{\mathbf{B}} = 1$ to show that $\frac{d\hat{\mathbf{B}}}{ds}$ is perpendicular to $\hat{\mathbf{B}}$.

(d) *Hence* prove the second Frenet-Serret formula $\frac{d\hat{\mathbf{B}}}{ds} = -\tau \hat{\mathbf{N}}$, where the torsion, τ, is some scalar function of s. [The torsion is a measure of how much the curve is twisting out of the plane of $\hat{\mathbf{T}}$ and $\mathbf{N}$.]

(e) Use the fact that $\hat{\mathbf{T}}$, $\hat{\mathbf{N}}$, $\hat{\mathbf{B}}$ form a mutually orthogonal right-handed triad of unit vectors, (from which it necessarily follows that $\hat{\mathbf{B}} = \hat{\mathbf{T}} \times \hat{\mathbf{N}}$, $\hat{\mathbf{T}} = \hat{\mathbf{N}} \times \hat{\mathbf{B}}$ and $\hat{\mathbf{N}} = \hat{\mathbf{B}} \times \hat{\mathbf{T}}$), to establish the third Frenet-Serret formula $\frac{d\hat{\mathbf{N}}}{ds} = \tau \hat{\mathbf{B}} - \kappa \hat{\mathbf{T}}$.

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6. The equation of a curve in $\mathbb{R}^3$ is given parametrically by

$$\vec{\mathbf{r}} = e^t \sin t \,\hat{\mathbf{i}} - \hat{\mathbf{j}} + e^t \cos t \,\mathbf{k}$$

Find $\vec{\mathbf{v}}, \vec{\mathbf{a}}, v, a_T, a_N, \kappa, \hat{\mathbf{T}}, \hat{\mathbf{N}}, \hat{\mathbf{B}}$.

Show that the curve lies entirely in one plane and find the equation of that plane.

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7. For the semi-cubical parabola $y^2 = x^3$,

(a) Write down the equation of the surface of revolution generated by rotating the upper half of this curve once about the x-axis.

(b) Write down a definite integral for the curved surface area of the surface of revolution from the origin to $x = c$ (where $c > 0$).

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8. The location of a particle at any time $t > 0$ is given in plane polar coordinates by

$$\vec{\mathbf{r}}(t) = r(t)\,\hat{\mathbf{r}}(t)$$

with the distance $r(t)$ and direction $\theta(t)$ given by $r(t) = 1 - e^{-t}$ and $\theta(t) = t$.

(a) Find the radial and transverse components of the velocity of the particle.
(b) Find the radial and transverse components of the acceleration of the particle.
(c) Describe the path of the particle in the steady state (as $t \to \infty$).
(d) Find the tangential and normal components of the acceleration of the particle.

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