ENGI 4430 Advanced Calculus for Engineering
Faculty of Engineering and Applied Science

Problem Set 7 Questions

[Line integrals, Green's theorem]

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1. Find the work done when an object is moved along the curve of intersection C of the circular paraboloid $z = x^2 + y^2$ and the plane $2x + y = 2$ from (1, 0, 1) to (0, 2, 4) by a force $\vec{\mathbf{F}} = \begin{bmatrix} \frac{2}{x} & \frac{1}{x} & 1 \end{bmatrix}^T$.

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2. Find the work done in travelling in $\mathbb{R}^2$ once anti-clockwise around the unit circle C, centered on the origin, in the presence of the force $\vec{\mathbf{F}} = \begin{bmatrix} x-y \\ x+y \end{bmatrix}$, using *both* of the following methods:

(a) by direct evaluation of the line integral $\oint_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}}$ and

(b) by using Green's theorem and evaluating $\iint_D \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) dA$.

(c) Is the vector field $\vec{\mathbf{F}}$ conservative?

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3. In class we saw that the line integral $\oint_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}}$ has the value 2π,

where $\vec{\mathbf{F}} = \begin{bmatrix} \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{bmatrix}^T$ and the path C is one anti-clockwise circuit of the unit circle centered on the origin.

(a) Show that the value of the line integral remains 2π when the vector field is replaced by

$$\vec{\mathbf{F}} = \begin{bmatrix} \frac{-y}{\left(\sqrt{x^2+y^2}\right)^n} & \frac{x}{\left(\sqrt{x^2+y^2}\right)^n} \end{bmatrix}^T, \quad (n \in \mathbb{Z}).$$

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3 (b) Show that the value of $\displaystyle\iint_D \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) dA$ (where D is the circular area enclosed by C) is ***not*** 2π unless $n < 2$.
Hint: transform the area integral into plane polar coordinates, with
$x = r \cos \theta$, $y = r \sin \theta$ and $dA = dx\, dy = r\, dr\, d\theta$.

(c) Parts (a) and (b) clearly show that Green's theorem is not applicable when $n \geq 2$. Why should Green's theorem not be used when $0 < n < 2$?

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4. Find a potential function for the vector field $\vec{\mathbf{F}} = \left[2xy\cos z \quad x^2 \cos z \quad -x^2 y \sin z \right]^{\mathrm{T}}$.
Use this potential function to evaluate the line integral $\displaystyle\int_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}}$ for any piecewise smooth simple curve C from the point (1, 1, –1) to the point (2, 0, 5).

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5. A thin wire of line density $\rho = (ax+b)e^z$ is laid out along a circle, centre the origin, radius r, in the x-y plane, (where a and b are any real constants and r is any positive real constant).

(a) Show that a requirement that all parts of the wire have positive mass leads to the constraint $b > |a| r$.

(b) Find the mass of the wire (as a function of a, b and r).

(c) Show that the centre of mass is at the point $\left(\dfrac{ar^2}{2b}, 0, 0 \right)$.

(d) With the requirement of part (a) in place, what are the maximum and minimum possible distances of the centre of mass from the origin?

(e) The **moment of inertia** I of a body about an axis of rotation L is defined by $\Delta I = r^2 \Delta m$ (where r is the distance of the element of mass Δm from the axis L). [The moment of inertia is thus a second moment.]
Find the moment of inertia of the thin wire about the z-axis.

(f) The **kinetic energy** E of a rigid body rotating with angular velocity ω about the axis L is given by $E = \frac{1}{2} I \omega^2$. Find the kinetic energy of the thin wire when it rotates about the z-axis with an angular velocity ω.

(g) The **angular momentum** (or moment of momentum) of a rigid body rotating with angular velocity ω about the axis L is $I\omega$. [In the absence of any friction or externally applied torque, the angular momentum is conserved.] Find the angular momentum of the thin wire when it rotates about the z-axis with an angular velocity ω.

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6. Find the mass and centre of mass of a thin wire that is stretched along a straight line between the origin and the point (6, 6, 6), given that the line density at (x, y, z) is $\dfrac{x+y+z}{100}\ \mathrm{kg\,m^{-1}}$.

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7. Find the potential function $V(x, y, z)$ for the vector field

$$\vec{\mathbf{F}} = x\left(6x\sin z + 2z\,e^{-x^2}\right)\hat{\mathbf{i}} + 4y\cos z\,\hat{\mathbf{j}} + \left(2x^3\cos z - 2y^2\sin z - e^{-x^2}\right)\hat{\mathbf{k}}\,.$$

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8. Find the work done by the force $\vec{\mathbf{F}} = xy\,\hat{\mathbf{i}} + y^2\,\hat{\mathbf{j}}$ in one circuit of the unit square, *without using Green's theorem*. [This is the lengthier solution to Example 8.08.]

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9. A wire is laid on the plane $z = 3$ in the shape of the arc of the parabola

$$y = x^2,\ z = 3$$

between the points $(0, 0, 3)$ and $(\sqrt{2}, 2, 3)$. Its line density is $\rho(x, y, z) = 6x$. Use the parametric form $\vec{\mathbf{r}}(t) = t\,\hat{\mathbf{i}} + t^2\,\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$, $\left(0 \le t \le \sqrt{2}\right)$ to find the exact mass of the wire.

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10. Evaluate $\displaystyle\oint_C \vec{\mathbf{F}}\bullet d\vec{\mathbf{r}}$, where $\vec{\mathbf{F}} = \begin{bmatrix} \dfrac{y}{x^2+y^2} & \dfrac{-x}{x^2+y^2} \end{bmatrix}^{\mathrm{T}}$ and C is the unit circle, centre at the origin. [Note that this is a counter-example, to demonstrate that potential functions can be ill-defined on non-simply connected domains]

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