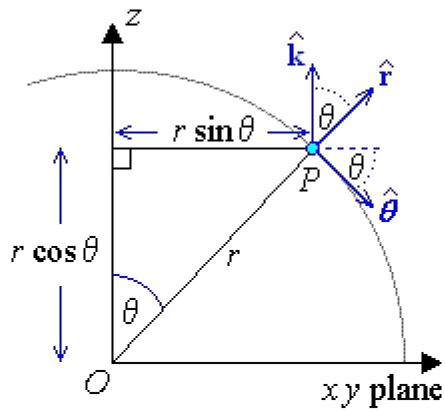
**Geometrical derivation of the Cartesian components of the spherical polar basis vectors**

Vertical plane containing $z$-axis and radial vector $\vec{\mathbf{r}}$:




$$\hat{\mathbf{r}} = (\hat{\mathbf{r}}\bullet\hat{\mathbf{i}})\hat{\mathbf{i}} + (\hat{\mathbf{r}}\bullet\hat{\mathbf{j}})\hat{\mathbf{j}} + (\hat{\mathbf{r}}\bullet\hat{\mathbf{k}})\hat{\mathbf{k}}$$

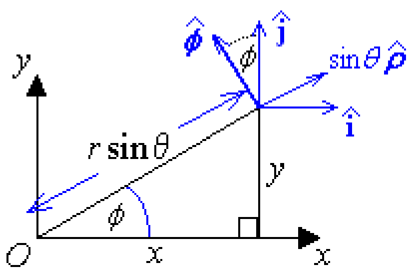
The projection of $\hat{\mathbf{r}}$ in the direction of the $z$ axis is obvious: the angle between $\hat{\mathbf{r}}$ and $\hat{\mathbf{k}}$ is $\theta$

$\Rightarrow \quad \hat{\mathbf{r}}\bullet\hat{\mathbf{k}} = 1\times1\times\cos\theta = \cos\theta$

The angle between $\hat{\boldsymbol{\theta}}$ and $\hat{\mathbf{k}}$ is $\theta + \frac{\pi}{2}$

$\Rightarrow \quad \hat{\boldsymbol{\theta}}\bullet\hat{\mathbf{k}} = 1\times1\times\cos\left(\theta + \frac{\pi}{2}\right) = -\sin\theta$

Equatorial plane ($\theta = 0$): The projection of $\hat{\mathbf{r}}$ onto the equatorial plane is $\sin\theta\,\hat{\boldsymbol{\rho}}$




The component of $\sin\theta\,\hat{\boldsymbol{\rho}}$ in the direction of the $x$ axis is $(\sin\theta\,\hat{\boldsymbol{\rho}})\bullet\hat{\mathbf{i}} = (\sin\theta\times1)\times1\times\cos\phi$

$\Rightarrow \quad \hat{\mathbf{r}}\bullet\hat{\mathbf{i}} = \sin\theta\cos\phi$

Similarly $\hat{\mathbf{r}}\bullet\hat{\mathbf{j}} = \sin\theta\sin\phi$

so that

$$\hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{i}} + \sin\theta\sin\phi\,\hat{\mathbf{j}} + \cos\theta\,\hat{\mathbf{k}}$$

The projection of $\hat{\boldsymbol{\theta}}$ onto the equatorial plane is $\cos\theta\,\hat{\boldsymbol{\rho}}$

The components of this vector in the $x$ and $y$ directions are similar to those for $\sin\theta\,\hat{\boldsymbol{\rho}}$

It soon follows that

$$\hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\hat{\mathbf{i}} + \cos\theta\sin\phi\,\hat{\mathbf{j}} - \sin\theta\,\hat{\mathbf{k}}$$

$\hat{\boldsymbol{\phi}}$ has an angle of $\left(\phi + \frac{\pi}{2}\right)$ with the $x$ axis, an angle of $\phi$ with the $y$ axis

and is orthogonal to the $z$ axis $\Rightarrow \quad \hat{\boldsymbol{\phi}}\bullet\hat{\mathbf{i}} = \cos\left(\phi + \frac{\pi}{2}\right) = -\sin\phi, \quad \hat{\boldsymbol{\phi}}\bullet\hat{\mathbf{j}} = \cos\phi, \quad \hat{\boldsymbol{\phi}}\bullet\hat{\mathbf{k}} = 0$

$\Rightarrow \quad \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{i}} + \cos\phi\,\hat{\mathbf{j}}$

This reproduces the three rows of the coordinate conversion matrix on page 7.04:

$$\mathrm{A} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix}$$