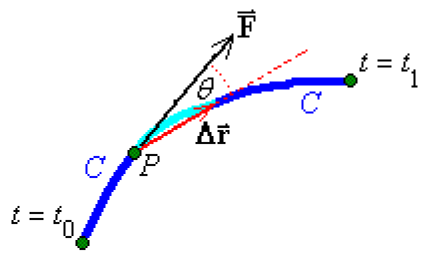
8. Line Integrals

Two applications of line integrals are treated here: the evaluation of work done on a particle as it travels along a curve in the presence of a [vector field] force; and the evaluation of the location of the centre of mass of a wire.

Work done:

The work done by a force $\vec{\mathbf{F}}$ in moving an elementary distance $\Delta\vec{\mathbf{r}}$ along a curve C is approximately the product of the component of the force in the direction of $\Delta\vec{\mathbf{r}}$ and the distance $|\ \Delta\vec{\mathbf{r}}\ |$ travelled:



Integrating along the curve C yields the total work done by the force $\vec{\mathbf{F}}$ in moving along the curve C:

$$W = \int_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}}$$

Example 8.01

Find the work done by $\vec{\mathbf{F}} = \begin{bmatrix} -y & x & z \end{bmatrix}^{\mathrm{T}}$ in moving once around the closed curve C (defined in parametric form by $x = \cos t$, $y = \sin t$, $z = 0$, $0 \le t < 2\pi$).

Example 8.01 (continued)

Example 8.02
Find the work done by $\vec{\mathbf{F}} = \left[x \;\; y \;\; z \right]^{\mathrm{T}}$ in moving around the curve C (defined in parametric form by $x = \cos t$, $y = \sin t$, $z = 0$, $0 \le t < 2\pi$).

If the initial and terminal points of a curve C are identical and the curve meets itself nowhere else, then the curve is said to be a **simple closed curve**.

Notation:

When C is a simple closed curve, write $\int_C \vec{\mathbf{F}}\cdot d\vec{\mathbf{r}}$ as $\oint_C \vec{\mathbf{F}}\cdot d\vec{\mathbf{r}}$.

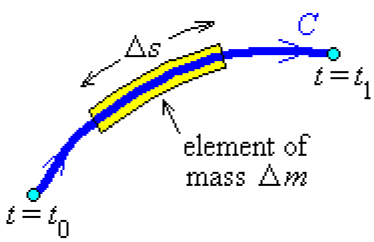
$\vec{\mathbf{F}}$ is a **conservative vector field** if and only if $\oint_C \vec{\mathbf{F}}\cdot d\vec{\mathbf{r}} = 0$ for all simple closed curves C in the domain.

Be careful of where the endpoints are and of the order in which they appear (the orientation of the curve). The identity $\int_{t_0}^{t_1} \vec{\mathbf{F}}\cdot\frac{d\vec{\mathbf{r}}}{dt}\,dt \equiv -\int_{t_1}^{t_0} \vec{\mathbf{F}}\cdot\frac{d\vec{\mathbf{r}}}{dt}\,dt$ leads to the result

$$\oint_{C\,\circlearrowright} \vec{\mathbf{F}}\cdot d\vec{\mathbf{r}} = -\oint_{C\,\circlearrowleft} \vec{\mathbf{F}}\cdot d\vec{\mathbf{r}} \quad \forall \text{ simple closed curves } C$$

Another Application of Line Integrals: **The Mass of a Wire**

Let C be a segment $(t_0 \le t \le t_1)$ of wire of line density $\rho(x,y,z)$. Then



First moments about the coordinate planes:

The location $\langle\vec{\mathbf{r}}\rangle$ of the centre of mass of the wire is $\langle\vec{\mathbf{r}}\rangle = \frac{\vec{\mathbf{M}}}{m}$, where the moment

$$\vec{\mathbf{M}} = \int_{t_0}^{t_1} \rho\,\vec{\mathbf{r}}\,\frac{ds}{dt}\,dt\,, \quad m = \int_{t_0}^{t_1} \rho\,\frac{ds}{dt}\,dt \quad \text{and} \quad \frac{ds}{dt} = \left|\frac{d\vec{\mathbf{r}}}{dt}\right| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,.$$

Example 8.03

Find the mass and centre of mass of a wire C (described in parametric form by $x = \cos t$, $y = \sin t$, $z = t$, $-\pi \le t \le \pi$) of line density $\rho = z^2$.

Example 8.03 (continued)

Green's Theorem

Some definitions:

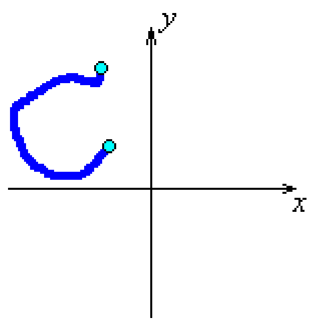
A curve C on $\mathbb{R}^2$ (defined in parametric form by $\vec{\mathbf{r}}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}}, \quad a \le t \le b$) is **closed** iff $(x(a), y(a)) = (x(b), y(b))$.

The curve is **simple** iff $\vec{\mathbf{r}}(t_1) \ne \vec{\mathbf{r}}(t_2)$ for all t_1, t_2 such that $a < t_1 < t_2 < b$;
(that is, the curve neither touches nor intersects itself, except possibly at the end points).

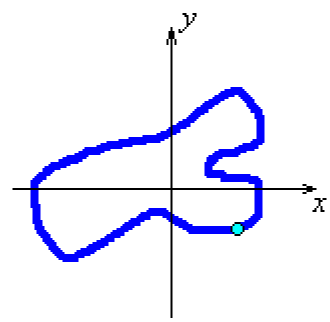
Example 8.04

Two simple curves:

open

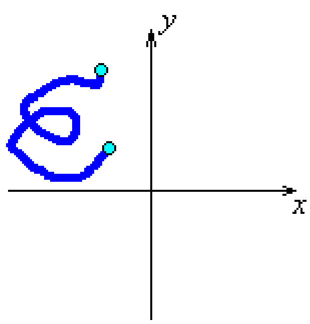


closed

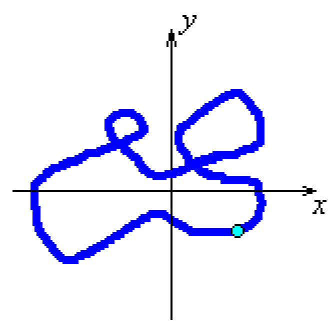


Two non-simple curves:

open



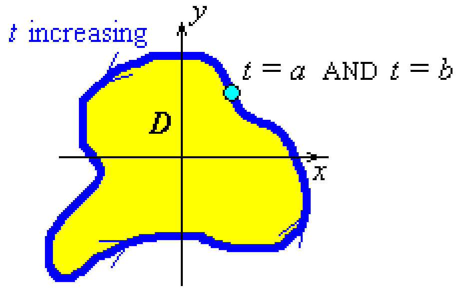
closed



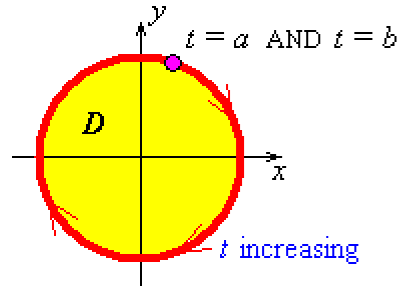
Orientation of closed curves:

A closed curve C has a positive orientation iff a point $\vec{\mathbf{r}}(t)$ moves around C in an anticlockwise sense as the value of the parameter t increases.

Example 8.05



Positive orientation



Negative orientation

Let D be the finite region of $\mathbb{R}^2$ bounded by C. When a particle moves along a curve with positive orientation, D is always to the left of the particle.

For a simple closed curve C enclosing a finite region D of $\mathbb{R}^2$ and for any vector function $\vec{\mathbf{F}} = \begin{bmatrix} f_1 & f_2 \end{bmatrix}^T$ that is differentiable everywhere on C and everywhere in D, **Green's theorem** is valid:

$$\oint_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \iint_D \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) dA$$

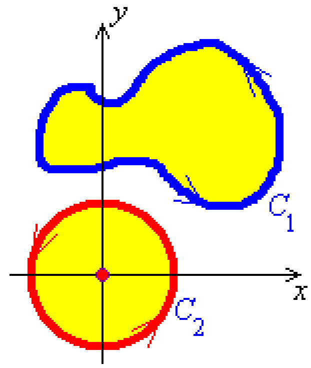
The region D is entirely in the xy-plane, so that the unit normal vector everywhere on D is $\hat{\mathbf{k}}$. Let the differential vector $d\vec{\mathbf{A}} = dA\,\hat{\mathbf{k}}$, then Green's theorem can also be written as

$$\oint_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \iint_D \left(\vec{\nabla} \times \vec{\mathbf{F}} \right) \cdot \hat{\mathbf{k}}\, dA = \iint_D \left(\operatorname{curl} \vec{\mathbf{F}} \right) \cdot d\vec{\mathbf{A}}$$

Green's theorem is valid if there are no singularities in D.
A [non-examinable] proof is provided at the end of this chapter.

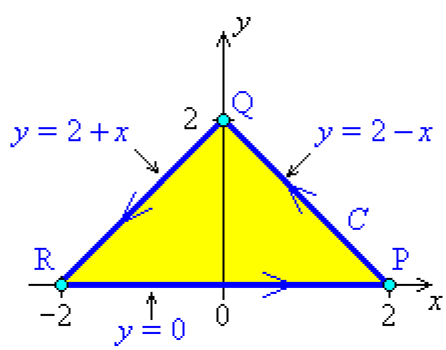
Example 8.06

$\vec{\mathbf{F}} = \begin{bmatrix} \dfrac{x}{r} & 0 \end{bmatrix}^{\mathrm{T}}$:



Example 8.07

For $\vec{\mathbf{F}} = \begin{bmatrix} x+y \\ x-y \end{bmatrix}$ and C as shown, evaluate $\oint_C \vec{\mathbf{F}} \bullet \mathbf{d}\vec{\mathbf{r}}$.

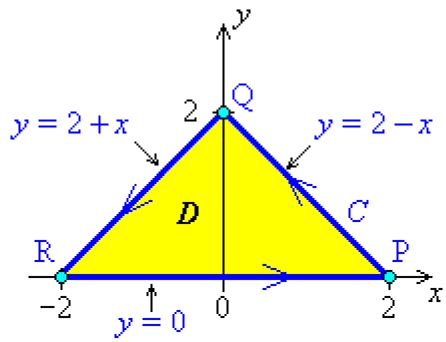


Example 8.07 (continued)

$$\vec{\mathbf{F}} = \begin{bmatrix} x+y \\ x-y \end{bmatrix}$$

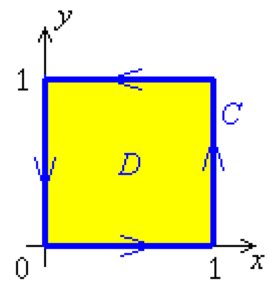
Example 8.07 (continued)

OR use Green's theorem!



Example 8.08

Find the work done by the force $\vec{\mathbf{F}} = xy\,\hat{\mathbf{i}} + y^2\,\hat{\mathbf{j}}$ in one circuit of the unit square.



Path Independence

Gradient Vector Fields:

If $\vec{\mathbf{F}} = \vec{\nabla} V$, then $\vec{\mathbf{F}} = \left[\frac{\partial V}{\partial x} \quad \frac{\partial V}{\partial y} \right]^{\mathrm{T}} \quad \Rightarrow$

Path Independence

If $\vec{\mathbf{F}} = \vec{\nabla} V \quad \left(\text{or } \vec{\mathbf{F}} = -\vec{\nabla} V\right)$, then V is a **potential function** for $\vec{\mathbf{F}}$.

Let the path C travel from point P_0 to point P_1:

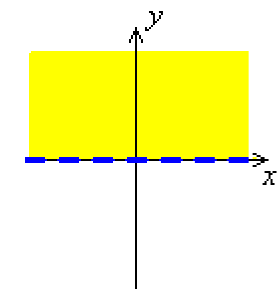
Domain

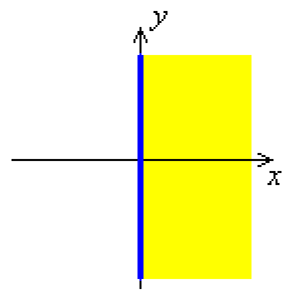
A region Ω of $\mathbb{R}^2$ is a **domain** if and only if

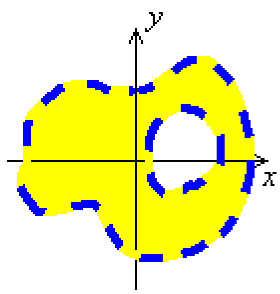
1) For all points P_0 in Ω, there exists a circle, centre P_0, all of whose interior points are inside Ω; and
2) For all points P_0 and P_1 in Ω, there exists a piecewise smooth curve C, entirely in Ω, from P_0 to P_1.

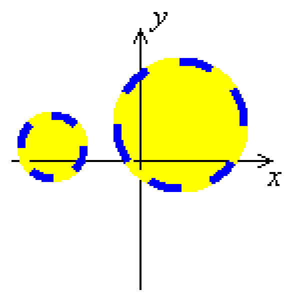
<u>Example 8.09</u> Are these domains?

$\{ (x, y) \mid y > 0 \}$ $\{ (x, y) \mid x \geq 0 \}$









If a domain is not specified, then, by default, it is assumed to be all of $\mathbb{R}^2$.

When a vector field $\vec{\mathbf{F}}$ is defined on a simply connected domain Ω, these statements are all equivalent (that is, **all** of them are true or all of them are false):

- $\vec{\mathbf{F}} = \vec{\nabla} V$ for some scalar field V that is differentiable everywhere in Ω;
- $\vec{\mathbf{F}}$ is conservative;
- $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ is path-independent (has the same value no matter which path within Ω is chosen between the two endpoints, for any two endpoints in Ω);
- $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = V_{\text{end}} - V_{\text{start}}$ (for any two endpoints in Ω);
- $\oint_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = 0$ for all closed curves C lying entirely in Ω;
- $\dfrac{\partial f_2}{\partial x} = \dfrac{\partial f_1}{\partial y}$ everywhere in Ω; and
- $\vec{\nabla} \times \vec{\mathbf{F}} = \vec{\mathbf{0}}$ everywhere in Ω (so that the vector field $\vec{\mathbf{F}}$ is irrotational).

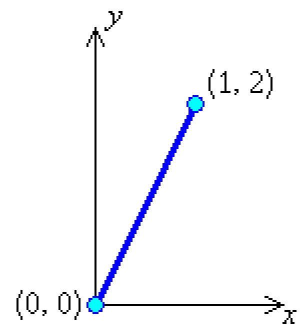
There must be no singularities anywhere in the domain Ω in order for the above set of equivalencies to be valid.

Example 8.10

Evaluate $\int_C \left((2x + y)\, dx + \left(x + 3y^2 \right) dy \right)$ where C is any piecewise-smooth curve from $(0, 0)$ to $(1, 2)$.

Example 8.10 by direct evaluation of the line integral

Let us pursue instead a particular path from (0, 0) to (1, 2).
The straight line path C_1 is a segment of the line
$y = 2x \quad \Rightarrow \quad x = \tfrac{1}{2} y$.

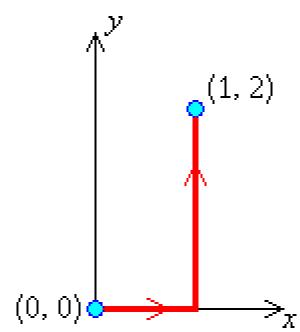


$$\Rightarrow \quad I = \int_{C_1} \left((2x + y)\, dx + \left(x + 3y^2 \right) dy \right) =$$

An alternative evaluation of $I = \int_{C_1} \vec{\mathbf{F}} \bullet \mathbf{d}\vec{\mathbf{r}}$ is to use x as the parameter in both integrals (that is, to express y in terms of x throughout). Then

$$\Rightarrow \quad I = \int_{C_1} \left((2x + y)\, dx + \left(x + 3y^2 \right) dy \right) =$$

An alternative path C_2 involves going round the other two sides of the triangle, first from (0, 0) horizontally to (1, 0) then from there vertically to (1, 2).
On the first leg $y \equiv 0 \quad \Rightarrow \quad dy \equiv 0$,
so that the second part of the integral vanishes.
On the second leg $x \equiv 1 \quad \Rightarrow \quad dx \equiv 0$,
so that the first part of the integral vanishes.

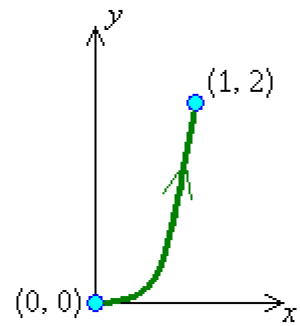


Therefore

$$I = \int_{C_2} \left((2x + y)\, dx + \left(x + 3y^2 \right) dy \right) =$$

Example 8.10 by direct evaluation of the line integral

Yet another possibility is C_3 an arc of the parabola $y = 2x^2$.



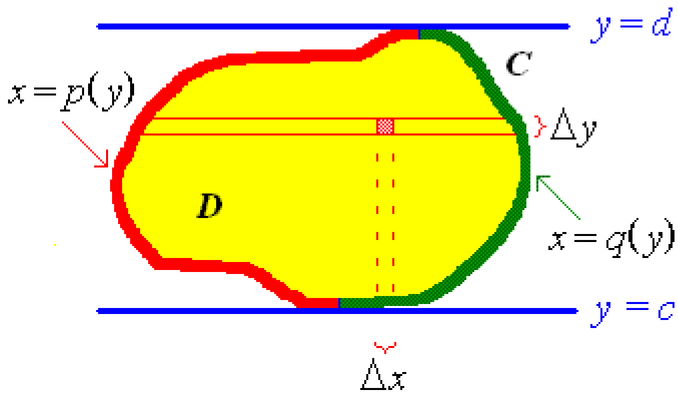
$$\Rightarrow \quad I = \int_{C_3} \left((2x + y)\, dx + \left(x + 3y^2\right) dy \right) =$$

Note that the above suggests that $I = \int_{(0,0)}^{(1,2)} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ might be path-independent, because evaluations along three different paths have all produced the same answer. But this is ***not*** a proof of path independence. For a proof, one must establish that $\vec{\mathbf{F}}$ is conservative, either by finding the potential function, or by showing that $\text{curl}\, \vec{\mathbf{F}} \equiv \vec{\mathbf{0}}$.

Outline of a Proof of Green's Theorem [not examinable]

Let $\vec{\mathbf{F}} = P(x,y)\,\hat{\mathbf{i}} + Q(x,y)\,\hat{\mathbf{j}}$.

Consider a convex region D as shown. Left and right boundaries can be identified.



Then

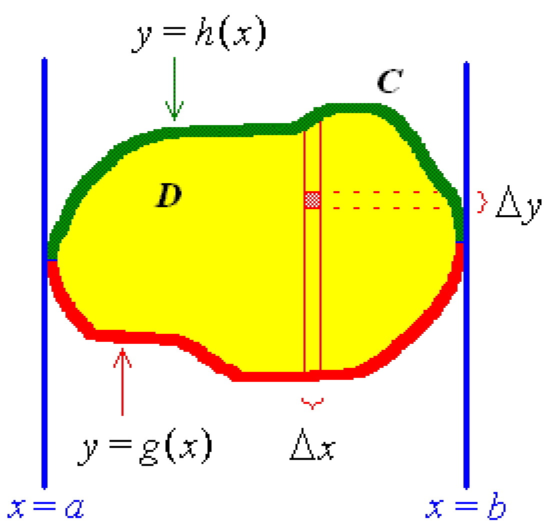
$$\iint_D \frac{\partial Q}{\partial x}\, dA = \int_c^d \int_{p(y)}^{q(y)} \frac{\partial Q}{\partial x}\, dx\, dy$$

$$= \int_c^d \Big[Q(x,y) \Big]_{x=p(y)}^{x=q(y)} dy$$

$$= \int_c^d \Big(Q\big(q(y),y\big) - Q\big(p(y),y\big) \Big)\, dy = \int_c^d Q\big(q(y),y\big)\, dy + \int_d^c Q\big(p(y),y\big)\, dy$$

But the path along $x = q(y)$ from $y = c$ to $y = d$ followed by the path along $x = p(y)$ from $y = d$ back to $y = c$ constitutes one complete circuit around the closed path C.

$$\Rightarrow \quad \iint_D \frac{\partial Q}{\partial x}\, dA = \oint_C Q\, dy$$



Lower and upper boundaries for the region can also be identified.

$$\iint_D \frac{\partial P}{\partial y}\, dA = \int_a^b \int_{g(x)}^{h(x)} \frac{\partial P}{\partial y}\, dy\, dx$$

$$= \int_a^b \Big[P(x,y) \Big]_{y=g(x)}^{y=h(x)} dx$$

$$= \int_a^b \Big(P\big(x,h(x)\big) - P\big(x,g(x)\big) \Big)\, dx$$

$$= -\int_b^a P\big(x,h(x)\big)\, dx - \int_a^b P\big(x,g(x)\big)\, dx$$

But the path along $y = g(x)$ from $x = a$ to $x = b$ followed by the path along $y = h(x)$ from $x = b$ back to $x = a$ constitutes one complete circuit around the closed path C.

$$\Rightarrow \quad \iint_D \frac{\partial P}{\partial y}\, dA = -\oint_C P\, dx \quad \Rightarrow \quad \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \oint_C \left(P\, dx + Q\, dy \right)$$

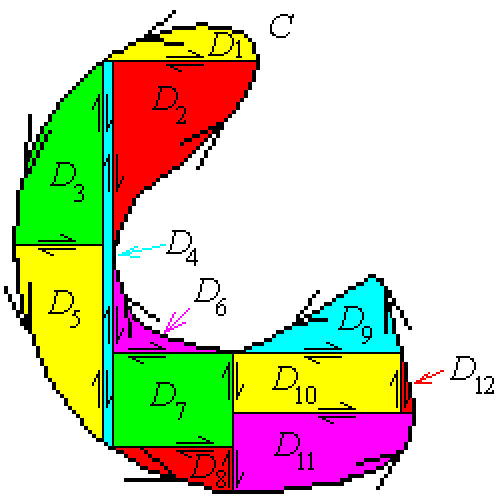
Green's Theorem (continued)

But $\vec{\mathbf{F}}\bullet d\vec{\mathbf{r}} = \begin{bmatrix} P \\ Q \end{bmatrix} \bullet \begin{bmatrix} dx \\ dy \end{bmatrix} = P\,dx + Q\,dy$

Therefore

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \oint_C \vec{\mathbf{F}}\bullet d\vec{\mathbf{r}}$$

This proof can be extended to non-convex regions. Simply divide them up into convex sub-regions and apply Green's theorem to each sub-region.



The line integrals along common interior boundaries cancel out because they are travelled in opposite directions along the same line. The boundary of each convex sub-region D_i is a simple closed curve C_i, for which Green's theorem is valid:

$$\oint_{C_i} \vec{\mathbf{F}}\bullet d\vec{\mathbf{r}} = \iint_{D_i} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

$$\Rightarrow \quad \sum_{\forall i} \oint_{C_i} \vec{\mathbf{F}}\bullet d\vec{\mathbf{r}} = \sum_{\forall i} \iint_{D_i} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

Therefore Green's theorem is also valid for any simply-connected region.

[End of Chapter 8]

[Space for Additional Notes]