10. Gauss' Divergence Theorem

Let S be a piecewise-smooth closed surface enclosing a volume V in $\mathbb{R}^3$ and let $\vec{\mathbf{F}}$ be a vector field. Then

the net flux of $\vec{\mathbf{F}}$ out of V is $\Phi = \oiint_S \vec{\mathbf{F}} \bullet \mathbf{d}\vec{\mathbf{S}} = \oiint_S F_N \, dS$,

where F_N is the component of $\vec{\mathbf{F}}$ normal to the surface S.

But the divergence of $\vec{\mathbf{F}}$ is a flux density, or an "outflow per unit volume" at a point.

Integrating div $\vec{\mathbf{F}}$ over the entire enclosed volume must match the net flux out through the boundary S of the volume V. **Gauss' divergence theorem** then follows:

$$\oiint_S \vec{\mathbf{F}} \bullet \mathbf{d}\vec{\mathbf{S}} = \iiint_V \vec{\nabla} \bullet \vec{\mathbf{F}} \, dV$$

Example 10.01 (Example 9.08 repeated)

Find the total flux Φ of the vector field $\vec{\mathbf{F}} = z\hat{\mathbf{k}}$ through the simple closed surface S

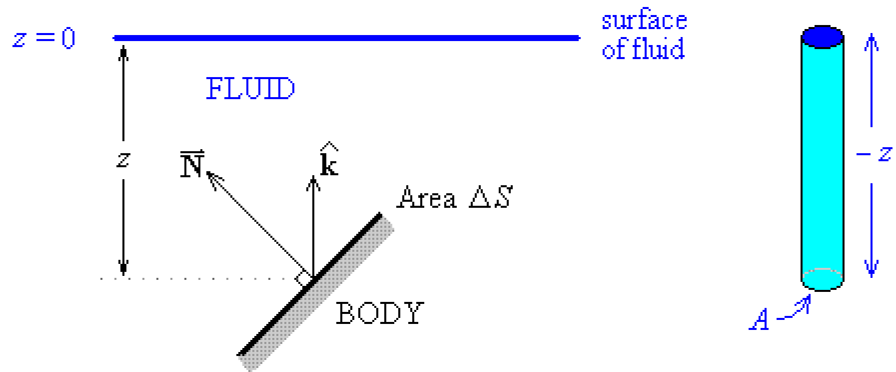
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Use Gauss' Divergence Theorem: $\oiint_S \vec{\mathbf{F}} \bullet \mathbf{d}\vec{\mathbf{S}} = \iiint_V \text{div}\, \vec{\mathbf{F}} \, dV$

$\vec{\mathbf{F}}$ is differentiable everywhere in $\mathbb{R}^3$, so Gauss' divergence theorem is valid.

Example 10.02 **Archimedes' Principle**

Gauss' divergence theorem may be used to derive Archimedes' principle for the buoyant force on a body totally immersed in a fluid of constant density ρ (independent of depth). Examine an elementary section of the surface S of the immersed body, at a depth $z < 0$ below the surface of the fluid:



The pressure at any depth z is the weight of fluid per unit area from the column of fluid above that area.

pressure = p =

The normal vector $\vec{\mathbf{N}}$ to S is directed outward, but the hydrostatic force on the surface (due to the pressure p) acts inward. The element of hydrostatic force on ΔS is

The element of buoyant force on ΔS is the component of the hydrostatic force in the direction of $\hat{\mathbf{k}}$ (vertically upwards):

Define $\vec{\mathbf{F}} = \rho g z \hat{\mathbf{k}}$ and $\mathbf{d}\vec{\mathbf{S}} = \hat{\mathbf{N}}\, dS$.
Summing over all such elements ΔS, the total buoyant force on the immersed object is

Example 10.02 Archimedes' Principle (continued)

Therefore the total buoyant force on an object fully immersed in a fluid equals the weight of the fluid displaced by the immersed object (Archimedes' principle).

Gauss' Law

A point charge q at the origin O generates an electric field

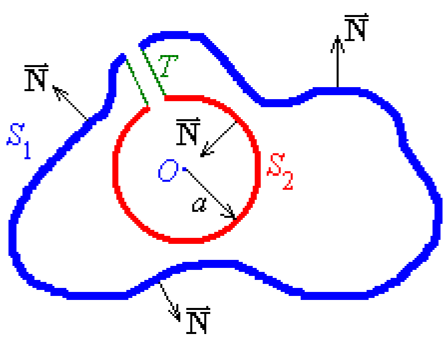
$$\vec{\mathbf{E}} = \frac{q}{4\pi\varepsilon r^3}\vec{\mathbf{r}} = \frac{q}{4\pi\varepsilon r^2}\hat{\mathbf{r}}$$

If S is a smooth simple closed surface **not** enclosing the charge, then the total flux through S is

If S does enclose the charge, then one cannot use Gauss' divergence theorem, because

Remedy:

Construct a surface S_1 identical to S except for a small hole cut where a narrow tube T connects it to another surface S_2, a sphere of radius a centre O and entirely inside S. Let $S^* = S_1 \cup T \cup S_2$ (which is a simple closed surface), then



Gauss' Law (continued)

Gauss' Law (continued)

Gauss' law for the net flux through any smooth simple closed surface *S*, in the presence of a point charge *q* at the origin, then follows:

$$\oint\!\!\!\oint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{S}} = \begin{cases} \dfrac{q}{\varepsilon} & \text{if } S \text{ encloses } O \\ 0 & \text{otherwise} \end{cases}$$

Example 10.03 **Poisson's Equation**

The exact location of the enclosed charge is immaterial, provided it is somewhere inside the volume V enclosed by the surface S. The charge therefore does not need to be a concentrated point charge, but can be spread out within the enclosed volume V. Let the charge density be $\rho(x, y, z)$, then the total charge enclosed by S is

Stokes' Theorem

Let $\vec{\mathbf{F}}$ be a vector field acting parallel to the *xy*-plane. Represent its Cartesian components by $\vec{\mathbf{F}} = f_1\hat{\mathbf{i}} + f_2\hat{\mathbf{j}} = \begin{bmatrix} f_1 & f_2 & 0 \end{bmatrix}^{\mathrm{T}}$. Then

$$\vec{\nabla}\times\vec{\mathbf{F}} = \begin{vmatrix} \hat{\mathbf{i}} & \dfrac{\partial}{\partial x} & f_1 \\ \hat{\mathbf{j}} & \dfrac{\partial}{\partial y} & f_2 \\ \hat{\mathbf{k}} & \dfrac{\partial}{\partial z} & 0 \end{vmatrix} \quad \Rightarrow \quad \left(\vec{\nabla}\times\vec{\mathbf{F}}\right)\bullet\hat{\mathbf{k}} = \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}$$

Green's theorem can then be expressed in the form

$$\oint_C \vec{\mathbf{F}}\bullet d\vec{\mathbf{r}} = \iint_D \vec{\nabla}\times\vec{\mathbf{F}}\bullet\hat{\mathbf{k}}\, dA$$

Now let us twist the simple closed curve *C* and its enclosed surface out of the *xy*-plane, so that the unit normal vector $\hat{\mathbf{k}}$ is replaced by a more general normal vector $\vec{\mathbf{N}}$.
If the surface *S* (that is bounded in $\mathbb{R}^3$ by the simple closed curve *C*) can be represented by $z = f(x, y)$, then a normal vector at any point on *S* is

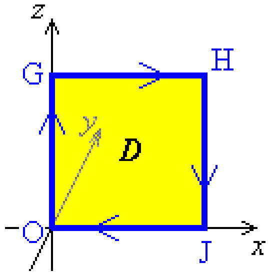
C is oriented coherently with respect to *S* if, as one travels along *C* with $\vec{\mathbf{N}}$ pointing from one's feet to one's head, *S* is always on one's left side. The resulting generalization of Green's theorem is **Stokes' theorem**:

$$\oint_C \vec{\mathbf{F}}\bullet d\vec{\mathbf{r}} = \iint_S \vec{\nabla}\times\vec{\mathbf{F}}\bullet\mathbf{N}\, dS = \iint_S \left(\mathrm{curl}\, \vec{\mathbf{F}}\right)\bullet d\vec{\mathbf{S}}$$

This can be extended further, to a non-flat surface *S* with a non-constant normal vector $\vec{\mathbf{N}}$.

Example 10.04

Find the circulation of $\vec{\mathbf{F}} = \begin{bmatrix} xyz & xz & e^{xy} \end{bmatrix}^{\mathrm{T}}$ around C : the unit square in the xz-plane.



Example 10.04 (continued)

Domain

A region Ω of $\mathbb{R}^3$ is a **domain** if and only if

1) For all points P_o in Ω, there exists a sphere, centre P_o, all of whose interior points are inside Ω; and
2) For all points P_o and P_1 in Ω, there exists a piecewise smooth curve C, entirely in Ω, from P_o to P_1.

A domain is **simply connected** if it "has no holes".

Example 10.05 Are these regions simply-connected domains?

The interior of a sphere.

The interior of a torus.

The first octant.

On a simply-connected domain the following statements are either all true or all false:

- $\vec{\mathbf{F}}$ is conservative.
- $\vec{\mathbf{F}} \equiv \vec{\nabla}\phi$
- $\vec{\nabla} \times \vec{\mathbf{F}} \equiv \vec{\mathbf{0}}$
- $\int_C \vec{\mathbf{F}} \bullet \mathbf{d}\vec{\mathbf{r}} = \phi(P_{\text{end}}) - \phi(P_{\text{start}})$ - independent of the path between the two points.
- $\oint_C \vec{\mathbf{F}} \bullet \mathbf{d}\vec{\mathbf{r}} = 0 \quad \forall C \subset \Omega$

Example 10.06

Find a potential function $\phi(x, y, z)$ for the vector field $\vec{\mathbf{F}} = \begin{bmatrix} 2x & 2y & 2z \end{bmatrix}^{\mathrm{T}}$.

First, check that a potential function exists at all:

$$\operatorname{curl} \vec{\mathbf{F}} = \vec{\nabla} \times \vec{\mathbf{F}} = \begin{vmatrix} \hat{\mathbf{i}} & \dfrac{\partial}{\partial x} & 2x \\ \hat{\mathbf{j}} & \dfrac{\partial}{\partial y} & 2y \\ \hat{\mathbf{k}} & \dfrac{\partial}{\partial z} & 2z \end{vmatrix} =$$

Example 10.06 (continued)

Example 10.07

Find a potential function for $\vec{\mathbf{F}} = e^y \, \hat{\mathbf{i}} + \left(x e^y + z^2 \right) \hat{\mathbf{j}} + 2yz \, \hat{\mathbf{k}}$ that has the value 1 at the origin.

Maxwell's Equations (***not*** examinable in this course)

We have seen how Gauss' and Stokes' theorems have led to Poisson's equation, relating the electric intensity vector $\vec{\mathbf{E}}$ to the electric charge density ρ:

$$\vec{\nabla} \bullet \vec{\mathbf{E}} = \frac{\rho}{\varepsilon}$$

Where the permittivity is constant, the corresponding equation for the electric flux density $\vec{\mathbf{D}}$ is one of Maxwell's equations: $\boxed{\vec{\nabla} \bullet \vec{\mathbf{D}} = \rho}$.

Another of Maxwell's equations follows from the absence of isolated magnetic charges (no magnetic monopoles): $\vec{\nabla} \bullet \vec{\mathbf{H}} = 0 \quad \Rightarrow \boxed{\vec{\nabla} \bullet \vec{\mathbf{B}} = 0}$, where $\vec{\mathbf{H}}$ is the magnetic intensity and $\vec{\mathbf{B}}$ is the magnetic flux density.

Faraday's law, connecting electric intensity with the rate of change of magnetic flux density, is $\oint_C \vec{\mathbf{E}} \bullet \mathbf{d}\vec{\mathbf{r}} = -\frac{\partial}{\partial t} \iint_S \vec{\mathbf{B}} \bullet \mathbf{d}\vec{\mathbf{S}}$. Applying Stokes' theorem to the left side produces

$$\boxed{\vec{\nabla} \times \vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t}}$$

Ampère's circuital law, $I = \oint_C \vec{\mathbf{H}} \bullet \mathbf{d}\vec{\mathbf{l}}$, leads to $\vec{\nabla} \times \vec{\mathbf{H}} = \vec{\mathbf{J}} + \vec{\mathbf{J}}_d$, where

the current density is $\vec{\mathbf{J}} = \sigma \vec{\mathbf{E}} = \rho_V \vec{\mathbf{v}}$, σ is the conductivity, ρ_V is the volume charge density; and the displacement charge density is $\vec{\mathbf{J}}_d = \frac{\partial \vec{\mathbf{D}}}{\partial t}$

The fourth Maxwell equation is

$$\boxed{\vec{\nabla} \times \vec{\mathbf{H}} = \vec{\mathbf{J}} + \frac{\partial \vec{\mathbf{D}}}{\partial t}}$$

The four Maxwell's equations together allow the derivation of the equations of propagating electromagnetic waves.

[End of Chapter 10]