# ENGI 9420 Engineering Analysis
# Assignment 5 Questions

2012 Fall

due in class on 2012 November 05 (Monday)

[Stability analysis, gradient operators; Chapters 4 and 5]

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1. Use Liénard's theorem to determine the stability of the solutions of the equation [10]

$$\frac{d^2x}{dt^2} + \left(x^2 - \frac{1}{3}\right)\frac{dx}{dt} + x^5 = 0$$

[Note that the associated linear system has a zero eigenvalue, which is an indeterminate form.]

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2. Find all limit cycles of the system of differential equations [10]

$$\frac{dx}{dt} = x - x^3 - xy^2 , \quad \frac{dy}{dt} = y - y^3 - yx^2$$

Hints: Compute $\frac{d}{dt}\left(x^2 + y^2\right)$; a limit cycle must be the orbit of a periodic solution that passes through no critical points of the system and encloses a critical point.]

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3. Use the Poincaré-Bendixon theorem to prove the existence of a non-trivial periodic solution of the differential equation [10]

$$\frac{d^2z}{dt^2} + \left(z^2 - 1\right)\frac{dz}{dt} + 2\left(\frac{dz}{dt}\right)^3 + z = 0$$

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4. Show that the system of differential equations [10]

$$\frac{dx}{dt} = x^3 - xy^2 - x^2 + y , \quad \frac{dy}{dt} = 2x^2y + y^3 + 2xy + y + 1$$

has no non-trivial periodic solution.

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5. Show that the system of differential equations [10]

$$\frac{dx}{dt} = 5x - xy^2 - y^3 , \quad \frac{dy}{dt} = 4y - x^2y + x^3$$

has no non-trivial periodic solution entirely inside the circle $x^2 + y^2 = 9$.

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6. For the vector field $\vec{\mathbf{F}} = e^{-kr}\,\vec{\mathbf{r}}$, where $\vec{\mathbf{r}} = \left[\, x\ y\ z \,\right]^{\mathrm{T}}$ and $k$ is a positive constant,

(a) Find the curl of $\vec{\mathbf{F}}$ in Cartesian coordinates. [3]

(b) Find the curl of $\vec{\mathbf{F}}$ while remaining in spherical polar coordinates throughout. [2]

(c) Work in Cartesian coordinates to find the divergence of $\vec{\mathbf{F}}$ as a function of $r$. [2]

(d) Work in spherical polar coordinates to find the divergence of $\vec{\mathbf{F}}$ as a function of $r$. [3]

(e) find where div $\vec{\mathbf{F}} = 0$ and classify this surface. [2]

(f) find where the magnitude $F$ of the vector field $\vec{\mathbf{F}}$ attains its maximum value. [2]

(g) show that $V(r) = \dfrac{-(1+kr)}{k^2}e^{-kr}$ is a potential function for the vector field $\vec{\mathbf{F}}$. [3]

(h) find the work done to move a particle from the origin to a place where div $\vec{\mathbf{F}} = 0$. [3]

[Note that work done is $\displaystyle\int_C \vec{\mathbf{F}}\bullet d\vec{\mathbf{r}} = \int_{t_0}^{t_1} \vec{\mathbf{F}}\bullet\frac{d\vec{\mathbf{r}}}{dt}\,dt$ .]

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7. A cylindrical parabolic coordinate system $(u, v, w)$ is defined by

$$x = uv\,, \quad y = \frac{u^2 - v^2}{2}\,, \quad z = w$$

(a) Find the Jacobian $\dfrac{\partial(x, y, z)}{\partial(u, v, w)} = \text{abs}\left(\det\begin{pmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} & \dfrac{\partial z}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} & \dfrac{\partial z}{\partial v} \\ \dfrac{\partial x}{\partial w} & \dfrac{\partial y}{\partial w} & \dfrac{\partial z}{\partial w} \end{pmatrix}\right)$ [3]

that allows the conversion of a volume differential $dV$ from Cartesian coordinates to these cylindrical polar coordinates.

(b) Find the scale factors $h_u, h_v, h_w$ for this cylindrical parabolic coordinate system. [3]

(c) Let the unit vectors of the cylindrical parabolic coordinate system be $\hat{\mathbf{u}}, \hat{\mathbf{v}}, \hat{\mathbf{w}}$. [3]
Find an expression for the gradient in this cylindrical parabolic coordinate system.

(d) Find an expression for the Laplacian $\nabla^2 f$ in this cylindrical parabolic coordinate system.
[3]

(e) Sketch on the same *x*-*y* plane any three members of each of the two families [3]
of coordinate curves $u$ = constant and $v$ = constant .

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8. The location of a particle at any time $t > 0$ is given in cylindrical polar coordinates by

$$\rho(t) = 2 - e^{-t}, \quad \phi(t) = t, \quad z(t) = e^{-t}.$$

Find the acceleration $\vec{\mathbf{a}}(t)$ in cylindrical polar coordinates. [5]

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9. The location of a particle at any time $t > 0$ is given in spherical polar coordinates by

$$r(t) = 2, \quad \theta(t) = \frac{\pi}{6}, \quad \phi(t) = t.$$

(a) Find the velocity $\vec{\mathbf{v}}(t)$ and speed $v(t)$ in spherical polar coordinates. [4]

(b) Find the acceleration $\vec{\mathbf{a}}(t)$ in spherical polar coordinates. [3]

(c) Describe the motion of the particle (what sort of path does it follow?) [3]

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☜ Return to the index of assignments On to the solutions ☞