ENGI 9420 Engineering Analysis
Assignment 6 Questions

2012 Fall

due in class on 2012 November 19 (Monday)

[Calculus of variations, Fourier series; Chapters 6 and 7]

Reminder of the principal concepts of the **calculus of variations**:

The path $y = f(x)$ between the points $(a, f(a))$ and $(b, f(b))$ (where both the x and y coordinates of both points are absolute constants) that minimizes (or maximizes) the value of the integral

$$I = \int_a^b F\left(x, y(x), y'(x)\right) dx$$

must satisfy the Euler equation for extremals, $\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0$

or, equivalently,

$$y'' \frac{\partial^2 F}{\partial y'^2} + y' \frac{\partial^2 F}{\partial y \, \partial y'} + \left(\frac{\partial^2 F}{\partial x \, \partial y'} - \frac{\partial F}{\partial y}\right) = 0$$

Special cases of this equation are:

If F is not explicitly dependent on y then $\frac{\partial F}{\partial y'} = c$

If F is not explicitly dependent on x then $y' \frac{\partial F}{\partial y'} - F = c$

If F is explicitly dependent on neither x nor y then

$$y = Ax + B \quad \left(\text{provided } \frac{\partial^2 F}{\partial y'^2} \not\equiv 0\right)$$

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1 (a) Find extremals $y(x)$ for $I = \int_{x_0}^{x_1} \frac{3 + (y')^2}{x^4} dx$ [7]

(b) Find the equation of the extremal that passes through the points (0, 3) and (1, 7). [5]

(c) Show that the extremal in part (b) minimizes the integral I. [8]

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2. Find the path between the points (1, 0) and (2, 1) along which a chain of minimum mass can be laid, when the density at every point on the chain is inversely proportional to the distance of that point from the *y* axis.
Classify (or describe) the geometric shape of this path.

Some key steps to the solution are provided here:

(a) Let ρ represent the line density (weight per unit length) of the chain. [5]

Then $\rho = \frac{k}{x}$, where k is a constant of proportionality.

Show that the total mass of the chain is $m = k\int_1^2 \frac{\sqrt{1+(y')^2}}{x}\,dx$.

This is the integral to be minimized.

(b) Show that the solution to the Euler-Lagrange equation for this integral is the ordinary differential equation [5]

$$\frac{y'}{x\sqrt{1+(y')^2}} = c_1$$

where c_1 is an arbitrary constant of integration.

(c) Show that the general solution to this ODE is [5]

$$y = \mp\frac{\sqrt{1-c_1^{\,2}x^2}}{c_1} + c_2$$

where c_2 is another arbitrary constant of integration.

(d) Use the fact that this path must pass through both of the points (1, 0) and (2, 1) to determine the values of the two arbitrary constants. This determines the path along which the chain must lie if it is to have an extremal value of mass among all chains that connect those two points. [5]

(e) Explain why this extremal path must provide the *minimum* value of mass (and not a maximum). [5]

(f) Inspect the functional form of your solution $y(x)$ and deduce the geometric shape of this minimal-mass path. [5]

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3. Find the Fourier series expansion of the function [15]

$$f(x) = x - x^3 \qquad (-1 \le x < 1)$$

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4. Find the Fourier cosine series expansion of the function [20]

$$f(x) = x - x^3 \qquad (0 < x < 1)$$

and

comment on the reasons for the differences in the approach to convergence of the series in questions (3) and (4) on the interval [0, 1].

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5. Two continuous functions $f(x), g(x)$ that are integrable on an interval $[a, b]$ are said to be orthogonal on that interval if and only if their inner product is zero:

$$(f, g) = \int_a^b f(x)\, g(x)\, dx = 0$$

(a) Show that any integrable even function is orthogonal to any integrable odd function on any interval that is symmetric about zero. [7]

(b) Find the relationship that the non-zero constants a and b must have if the odd function $f(x) = x$ is to be orthogonal to the odd function $g(x) = ax^3 + bx$ on the interval [–1, 1]. [8]

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