1. For the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial u}{\partial x\,\partial y} - 3\frac{\partial^2 u}{\partial y^2} = 4y$$

(a) Classify the partial differential equation as one of elliptic, parabolic or hyperbolic. [2]
(b) Find the general solution. [7]
(c) Find the complete solution, given the additional information [11]

$$u(0, y) = 0\,, \quad u_x(0, y) = y^2$$

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2 (a) Show that the only intersection of the curves $y = e^{-x^2}$ and $y = x$ must occur for some value of x in the interval $0 < x < 1$. [7]

(b) Use Newton's method with a reasonable initial value x_0 to estimate, correct to five decimal places, the value of x at which $f(x) = 0$, where [8]

$$f(x) = x - e^{-x^2}$$

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3. The non-linear second order ordinary differential equation

$$\frac{d^2 x}{dt^2} + (1 - x)\frac{dx}{dt} + 4x - x^2 = 0$$

can be represented by the system of first order ordinary differential equations

$$\dot{x} = y$$

$$\dot{y} = (x - 1)y - 4x + x^2$$

(a) Find the locations of both critical points. [4]
(b) For each critical point, identify its nature (node, centre, focus or saddle point) and stability. [7]
(c) Find the equations of the asymptotes for the linear approximation at any node or saddle point. [7]
[Note: the general solution is ***not*** required.]
(d) Sketch the phase portrait in the [linear] neighbourhood of each critical point. [6]
(e) Sketch the phase portrait for the non-linear system, including both critical points. [6]

BONUS QUESTION

(f) Find the equation of the separatrix (the curve that separates trajectories that terminate in a stable critical point from trajectories that recede to infinity). [+5]

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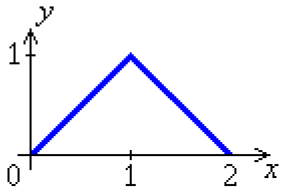
4. A perfectly elastic frictionless string is fixed at $x = 0$ and $x = 2$. It is stretched in a triangular configuration [15]



$$y(x,0) = f(x) = \begin{cases} x & (0 \le x < 1) \\ 2 - x & (1 \le x \le 2) \end{cases}$$

as illustrated and is released from rest.
The speed of waves on the string is $c = 6$.

Find a Fourier series expression for the subsequent displacement $y(x,t)$ of the string. You may quote

$$y(x,t) = \frac{2}{L}\sum_{n=1}^{\infty}\left(\int_0^L f(u)\sin\left(\frac{n\pi u}{L}\right)du\right)\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi c t}{L}\right).$$

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5. Find the path $y = f(x)$ between the points $(0,1)$ and $(1, 2e^3)$ that provides an extremum for the value of the integral [20]

$$I = \int_0^1 \left((y')^2 + 9y^2 + 12y\,e^{3x}\right)dx$$

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