1. Ordinary Differential Equations

An equation involving a function of one independent variable and the derivative(s) of that function is an ordinary differential equation (ODE).

The highest order derivative present determines the order of the ODE and the power to which that highest order derivative appears is the degree of the ODE. A general n^{th} order ODE is

$$F\left(x, y, y', y'', \ldots, y^{(n)}\right) = 0$$

Example 1.00.1

$\dfrac{d^2 y}{d x^2} + x\left(\dfrac{dy}{dx}\right)^2 = x^2 y$ is a second order first degree ODE.

Example 1.00.2

$x\left(\dfrac{dy}{dx}\right)^2 = x^2 y$ is a first order second degree ODE.

In this course we will usually consider first degree ODEs of first or second order only. The topics in this chapter are treated briefly, because it is assumed that graduate students will have seen this material during their undergraduate years.

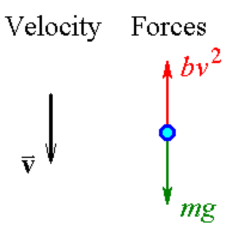
Sections in this Chapter:

1.01 First Order ODEs - Separation of Variables
1.02 Exact First Order ODEs
1.03 Integrating Factor
1.04 First Order Linear ODEs [+ Integration by Parts]
1.05 Bernoulli ODEs
1.06 Second Order Homogeneous Linear ODEs
1.07 Variation of Parameters
1.08 Method of Undetermined Coefficients
1.09 Laplace Transforms
1.10 Series Solutions of ODEs
1.11 The Gamma Function
1.12 Bessel and Legendre ODEs

1.01 First Order ODEs - Separation of Variables

Example 1.01.1

A particle falls under gravity from rest through a viscous medium such that the drag force is proportional to the square of the speed. Find the speed $v(t)$ at any time $t > 0$ and find the terminal speed v_∞.



The forces acting on the ball bearing are its weight downwards and friction upwards. Let m be the mass of the object, $g \approx 9.81 \text{ m s}^{-2}$ be the gravitational acceleration due to gravity.

Newton's second law of motion states

$$F = \frac{d}{dt}(mv) = m\frac{dv}{dt}$$

The ODE governing the motion follows:

$$m\frac{dv}{dt} = mg - bv^2 \quad \text{(Net force = weight − drag force)}$$

In standard form,

$$\underbrace{\left(bv^2 - mg\right)}_{M(t,v)} dt + \underbrace{(m)}_{N(t,v)} dv = 0$$

$M(t,v)$ ↑ $f(v)$ only; $N(t,v)$ ↑ const. $\quad \therefore$ type **separable**.

Whenever a first order ODE can be rewritten in the form

$$f(x)\,dx = g(y)\,dy$$

the method of separation of variables may be used.

The ODE in this problem may be separated into the form

$$\frac{m}{mg - bv^2}\,dv = dt \quad \Rightarrow \quad \frac{m}{-b\left(-\frac{mg}{b} + v^2\right)}\,dv = dt$$

Example 1.01.1 (continued)

$$\Rightarrow \quad \int \frac{dv}{v^2 - \frac{mg}{b}} = -\frac{b}{m}\int dt$$

$$\Rightarrow \quad \int \frac{dv}{v^2 - k^2} = -\frac{b}{m}\int dt \qquad \text{where} \quad k^2 = \frac{mg}{b}$$

Partial fractions:

$$\frac{1}{(v-k)(v+k)} = \frac{A}{v-k} + \frac{B}{v+k}$$

Using the "**cover-up rule**":

$$A = \frac{1}{\boxed{\cancel{(k-k)}}\,(k+k)} = \frac{1}{2k}$$

$$B = \frac{1}{(-k-k)\,\boxed{\cancel{(k+k)}}} = \frac{-1}{2k}$$

$$\Rightarrow \quad \frac{1}{v^2 - k^2} = \frac{1}{2k}\left(\frac{1}{v-k} - \frac{1}{v+k}\right)$$

$$\Rightarrow \quad \frac{1}{2k}\left(\ln(v-k) - \ln(v+k)\right) = -\frac{bt}{m} + C_1$$

$$\Rightarrow \quad \ln\left(\frac{v-k}{v+k}\right) = -\frac{2kbt}{m} + C_2 = -pt + C_2 \,,$$

$$\text{where} \quad p = \frac{2kb}{m} = \frac{2b}{m}\sqrt{\frac{mg}{b}} = 2\sqrt{\frac{bg}{m}}$$

$$\Rightarrow \quad \frac{v-k}{v+k} = e^{-pt + C_2} = A\,e^{-pt}$$

$$\Rightarrow \quad v - k = v\,A\,e^{-pt} + k\,A\,e^{-pt}$$

$$\Rightarrow \quad v\,(1 - A\,e^{-pt}) = k\,(1 + A\,e^{-pt})$$

Example 1.01.1 (continued)

General solution:

$$v(t) = \frac{k\left(1 + A e^{-pt}\right)}{1 - A e^{-pt}}$$

Initial condition: $v(0) = 0$

$$\Rightarrow \quad 0 = \frac{k(1+A)}{1-A} \quad \Rightarrow \quad A = -1$$

Complete solution:

$$v(t) = k \cdot \frac{1 - e^{-pt}}{1 + e^{-pt}}, \qquad \text{where} \quad k = \sqrt{\frac{mg}{b}} \quad \text{and} \quad p = 2\sqrt{\frac{bg}{m}}$$

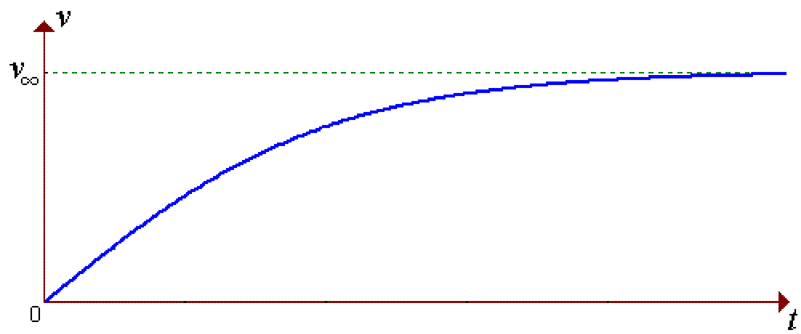
Terminal speed v_∞:

$$v_\infty = \lim_{t \to \infty} v(t) = k \frac{1-0}{1+0} = k = \sqrt{\frac{mg}{b}}$$

The terminal speed can also be found directly from the ODE.
At terminal speed, the acceleration is zero, so that the ODE simplifies to

$$m \frac{dv}{dt} = 0 = mg - b v_\infty^{\,2} \qquad \Rightarrow \quad v_\infty^{\,2} = \frac{mg}{b}$$

Graph of speed against time:



[For a 90 kg person in air, $b \approx 1$ kg m^{-1} $\rightarrow$ $k \approx 30$ ms^{-1} $\approx$ 100 km/h.
$v(t)$ is approximately linear at first, but air resistance builds quickly.
One accelerates to within 10 km/h of terminal velocity very fast, in just a few seconds.]

1.02 Exact First Order ODEs

If x and y are related implicitly by the equation $u(x, y) = c$ (constant), then the chain rule for differentiation leads to the ODE

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy = 0$$

Therefore, for the functions $M(x, y)$ and $N(x, y)$ in the first order ODE

$$M\,dx + N\,dy = 0 ,$$

if a **potential function** $u(x, y)$ exists such that

$$M = \frac{\partial u}{\partial x} \quad \text{and} \quad N = \frac{\partial u}{\partial y} ,$$

then $u(x, y) = c$ is the general solution to the ODE and the ODE is said to be **exact**.

Note that, for nearly all functions of interest, Clairault's theorem results in the identity

$$\frac{\partial^2 u}{\partial y\,\partial x} \equiv \frac{\partial^2 u}{\partial x\,\partial y}$$

This leads to a simple test to determine whether or not an ODE is exact:

$$\boxed{\frac{\partial M}{\partial y} \equiv \frac{\partial N}{\partial x} \quad \Rightarrow \quad M\,dx + N\,dy = 0 \quad \text{is exact}}$$

A separable first order ODE is also exact (after suitable rearrangement).

$$f(x)\,g(y)\,dx + h(x)\,k(y)\,dy = 0 \quad \text{[separable]}$$

$$\Rightarrow \quad \underbrace{\left(\frac{f(x)}{h(x)}\right)}_{M} dx + \underbrace{\left(\frac{k(y)}{g(y)}\right)}_{N} dy = 0$$

$$\Rightarrow \quad \frac{\partial M}{\partial y} = 0 = \frac{\partial N}{\partial x}$$

However, the converse is false. One counter-example will suffice.

Example 1.02.1

The ODE

$$\left(y\,e^{x} - x\right)dx + e^{x}dy = 0$$

is exact,

$$\left[M = y e^{x} - x\,, \quad N = e^{x} \qquad \Rightarrow \quad \frac{\partial M}{\partial y} = e^{x} = \frac{\partial N}{\partial x}\right]$$

but not separable.

To find the general solution, we seek a potential function $u(x, y)$ such that

$$\frac{\partial u}{\partial x} = y e^{x} - x \quad \text{and} \quad \frac{\partial u}{\partial y} = e^{x}$$

It does not take long to discover that

$$u = y e^{x} - \tfrac{1}{2}x^{2} = c$$

possesses the correct partial derivatives and is therefore the general solution of the ODE.

The solution may be expressed in explicit form as

$$y = \left(c + \tfrac{1}{2}x^{2}\right)e^{-x}$$

Example 1.02.2

Is the ODE $2y\,dx + x\,dy = 0$ exact?

$$M = 2y \quad \Rightarrow \quad \frac{\partial M}{\partial y} = 2\,, \qquad N = x \quad \Rightarrow \quad \frac{\partial N}{\partial x} = 1 \neq \frac{\partial M}{\partial y}$$

Therefore **NO**, the ODE is not exact.

Example 1.02.3

Is the ODE
$$A\left(2x^{2n+1}y^{n+1}\,dx \;+\; x^{2n+2}y^{n}\,dy\right) \;=\; 0$$
(where n is any constant and A is any non-zero constant) exact?
Find the general solution.

$$M \;=\; 2Ax^{2n+1}y^{n+1} \;\;\Rightarrow\;\; \frac{\partial M}{\partial y} \;=\; 2Ax^{2n+1}(n+1)\,y^{n}$$
$$N \;=\; Ax^{2n+2}y^{n} \;\;\Rightarrow\;\; \frac{\partial N}{\partial x} \;=\; A(2n+2)\,x^{2n+1}y^{n} \;=\; \frac{\partial M}{\partial y}$$
Therefore **YES**, the ODE is exact (for any *n* and for any non-zero *A*).

To find the general solution, we seek a potential function $u(x, y)$ such that
$$\frac{\partial u}{\partial x} \;=\; 2Ax^{2n+1}y^{n+1} \quad \text{and} \quad \frac{\partial u}{\partial y} \;=\; Ax^{2n+2}y^{n}$$
If $n \neq -1$ then $u \;=\; \dfrac{Ax^{2n+2}y^{n+1}}{n+1} \;=\; c_1$

If $n = -1$ then $u \;=\; A\ln\left(x^2 y\right) \;=\; c_1$

In either case, the general solution simplifies to $x^2 y = c$ or, in explicit form,
$$\boxed{y \;=\; \frac{c}{x^2}}$$

Note that the exact ODE in example 1.02.3 is just the non-exact ODE of example 1.02.2 multiplied by the factor $I(x, y) = Ax^{2n+1}y^{n}$. The ODEs are therefore equivalent and share the same general solution. The function $I(x, y) = Ax^{2n+1}y^{n}$ is an **integrating factor** for the ODE of example 1.02.2.

Also note that the integrating factor is not unique. In this case, *any* two distinct values of *n* generate two distinct integrating factors that both convert the non-exact ODE into an exact form. However, we need to guard against introducing a spurious singular solution $y \equiv 0$.

1.03 Integrating Factor

Occasionally it is possible to transform a non-exact first order ODE into exact form, using an integrating factor $I(x, y)$.
Suppose that

$$P\,dx + Q\,dy = 0$$

is not exact, but that

$$IP\,dx + IQ\,dy = 0$$

is exact.

Then, using the product rule,

$$M = I \cdot P \quad \Rightarrow \quad \frac{\partial M}{\partial y} = \frac{\partial I}{\partial y} P + I \frac{\partial P}{\partial y}$$

and

$$N = I \cdot Q \quad \Rightarrow \quad \frac{\partial N}{\partial x} = \frac{\partial I}{\partial x} Q + I \frac{\partial Q}{\partial x}$$

From the exactness condition

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \Rightarrow \quad \frac{\partial I}{\partial x} Q - \frac{\partial I}{\partial y} P = I \cdot \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \right)$$

This is an awkward partial differential equation. Where it is valid, we may use the assumption that the integrating factor is a function of x alone, to simplify its derivation.

$$\frac{dI}{dx} Q - 0 = I \cdot \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \right)$$

$$\Rightarrow \quad \frac{1}{I} \frac{dI}{dx} = \frac{1}{Q} \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \right)$$

This assumption is valid only if $\boxed{\frac{1}{Q} \cdot \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \right) = R(x)}$ is a function of x only.

If so, then the integrating factor is $\boxed{I(x) = e^{\int R(x)\,dx}}$

[Note that the arbitrary constant of integration can be omitted safely.] Then

$$u = \int M\,dx = \int e^{\int R(x)\,dx} \cdot P(x, y)\,dx, \quad \text{etc.}$$

Returning to

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \Rightarrow \quad \frac{\partial I}{\partial x} Q - \frac{\partial I}{\partial y} P = I \cdot \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \right)$$

If we assume that the integrating factor is a function of y alone, then

$$0 - \frac{dI}{dy} \cdot P = I \cdot \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \right) \quad \Rightarrow \quad \frac{1}{I} \cdot \frac{dI}{dy} = \frac{1}{P} \cdot \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)$$

This assumption is valid only if $\boxed{\frac{1}{P} \cdot \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) = S(y)}$ a function of y only.

If so, then the integrating factor is $\boxed{I(y) = e^{\int S(y)\,dy}}$ and

$$u = \int N\,dy = \int e^{\int S(y)\,dy} \cdot Q(x, y)\,dy, \quad \text{etc.}$$

Example 1.03.1 (Example 1.02.2 again)

Find the general solution of the ODE

$$2y\,dx + x\,dy = 0$$

$$P = 2y, \quad Q = x \quad \Rightarrow \quad \frac{1}{Q} \cdot \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \right) = \frac{2-1}{x} = \frac{1}{x} = R(x)$$

Therefore an integrating factor that is a function of x only does exist.

$$\int R(x)\,dx = \int \frac{1}{x}\,dx = \ln x \quad \Rightarrow \quad I(x) = e^{\int R(x)\,dx} = e^{\ln x} = x$$

Multiplying the original ODE by $I(x)$, we obtain the exact ODE

$$2xy\,dx + x^2\,dy = 0$$

To find the general solution, we seek a potential function $u(x, y)$ such that

$$\frac{\partial u}{\partial x} = 2xy \quad \text{and} \quad \frac{\partial u}{\partial y} = x^2$$

This leads quickly to the general solution, $u = x^2 y = c$ or, in explicit form,

$$\boxed{y = \frac{c}{x^2}}$$

Example 1.03.2

Find the general solution of the ODE

$$2xy\,dx + (2x^2 + 3y)\,dy = 0$$

Test for exactness:

$$P = 2xy \ , \quad Q = 2x^2 + 3y \quad \Rightarrow \quad \frac{\partial P}{\partial y} = 2x \ , \quad \frac{\partial Q}{\partial x} = 4x \neq \frac{\partial P}{\partial y}$$

Therefore the ODE is not exact.

Assume an integrating factor of the form $I(x)$:

$$\frac{1}{Q}\cdot\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = \frac{2x - 4x}{2x^2 + 3y} \neq \boldsymbol{R(x)}$$

Therefore an integrating factor that is a function of x only does *not* exist.

Assume an integrating factor of the form $I(y)$:

$$\frac{1}{P}\cdot\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = \frac{4x - 2x}{2xy} = \frac{1}{y} = R(y)$$

Therefore an integrating factor that is a function of y only *does* exist.

$$\int R(y)\,dy = \int \frac{1}{y}\,dy = \ln y \quad \Rightarrow \quad I(y) = e^{\int R(y)\,dy} = e^{\ln y} = y$$

Multiplying the original ODE by $I(y)$, we obtain the exact ODE

$$2xy^2\,dx + (2x^2y + 3y^2)\,dy = 0$$

To find the general solution, we seek a potential function $u(x, y)$ such that

$$\frac{\partial u}{\partial x} = 2xy^2 \quad \text{and} \quad \frac{\partial u}{\partial y} = 2x^2y + 3y^2$$

This leads quickly to the general solution, $u = \boxed{x^2y^2 + y^3 = c}$ or, in explicit form,

$$\boxed{x = \pm\frac{\sqrt{c - y^3}}{y}}$$

1.04 First Order Linear ODEs [+ Integration by Parts]

A special case of a first order ODE is the linear ODE:

$$\boxed{\frac{dy}{dx} + P(x)\,y = R(x)}$$

[or, in some cases,

$$\frac{dx}{dy} + Q(y)\,x = S(y)\ ]$$

Rearranging the first ODE into standard form,

$$(P(x)\,y - R(x))\,dx + 1\,dy = 0$$

Written in the standard exact form with a simple integrating factor in place, the ODE becomes

$$I(x)\,(P(x)\,y - R(x))\,dx + I(x)\,dy = 0$$

Compare this with the exact ODE

$$du = M(x, y)\,dx + N(x, y)\,dy = 0$$

The exactness condition $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow I(x)\cdot P(x) = \frac{dI}{dx} \Rightarrow \int \frac{dI}{I} = \int P\,dx$

Let $h(x) = \int P\,dx$, then $\ln I(x) = h(x)$

and the integrating factor is

$$I(x) = e^{h(x)}, \quad \text{where} \quad h(x) = \int P(x)\,dx \quad \left(\Rightarrow \frac{dh}{dx} = P(x)\right).$$

The ODE becomes the exact form

$$e^h\,(Py - R)\,dx + e^h\,dy = 0$$

Seek a potential function $u(x, y)$ such that

$$\frac{\partial u}{\partial x} = e^h\left(h'y - R\right) \quad \text{and} \quad \frac{\partial u}{\partial y} = e^h$$

$$\Rightarrow \quad u = y\,e^h - \int e^h R\,dx = C \qquad \Rightarrow \quad y\,e^h = \int e^h R\,dx + C$$

Therefore the general solution of $\frac{dy}{dx} + P(x)\,y = R(x)$ is

$$\boxed{y(x) = e^{-h(x)}\left(\int e^{h(x)} R(x)\,dx + C\right), \quad \text{where} \quad h(x) = \int P(x)\,dx}$$

Example 1.04.1

Solve the ordinary differential equation

$$\frac{dy}{dx} + \frac{2}{x}y = 1$$

This ODE is linear, with $P(x) = \frac{2}{x}$ and $R(x) = 1$.

$$h = \int P\,dx = \int \frac{2}{x}\,dx = 2\ln x = \ln\left(x^2\right)$$

The integrating factor is therefore $e^h = x^2$.

$$\int e^h R\,dx = \int x^2 1\,dx = \frac{x^3}{3}$$

$$\Rightarrow \quad y = e^{-h}\left(\int e^h R\,dx + C\right) = \frac{1}{x^2}\left(\frac{x^3}{3} + C\right)$$

The general solution is therefore

$$\boxed{y = \frac{x}{3} + \frac{C}{x^2}}$$

Alternative methods:

The ODE is not separable.

Re-arrange the ODE into the form

$$(2y - x)\,dx + x\,dy = 0$$

$$P = 2y - x \quad \text{and} \quad Q = x \qquad \Rightarrow \quad P_y = 2\,, \quad Q_x = 1 \neq P_y$$

This ODE is not exact.

$$\frac{P_y - Q_x}{Q} = \frac{2-1}{x} = \frac{1}{x} = R(x)$$

$$\Rightarrow \quad \int R\,dx = \int \frac{1}{x}\,dx = \ln x \qquad \Rightarrow \quad I(x) = x$$

Example 1.04.1 (continued)

The exact ODE is therefore

$(2xy - x^2)\, dx + x^2\, dy = 0$

$$\frac{\partial u}{\partial x} = 2xy - x^2 \quad \text{and} \quad \frac{\partial u}{\partial y} = x^2 \qquad \Rightarrow \quad u = x^2 y - \frac{x^3}{3} = c$$

The same explicit solution then follows:

$$\boxed{y = \frac{x}{3} + \frac{C}{x^2}}$$

OR

Try to re-write the ODE in another exact form $\frac{d}{dx}\big(u(x,y)\big) = v(x)$:

$$\frac{dy}{dx} + \frac{2}{x} y = 1 \qquad \Rightarrow \quad x^2 \frac{dy}{dx} + 2xy = x^2 \quad \Rightarrow \quad \frac{d}{dx}\left(x^2 y\right) = x^2$$

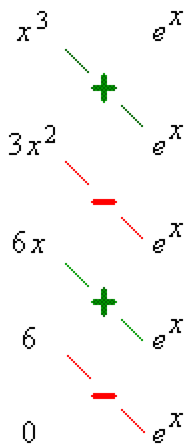
$$\Rightarrow \quad x^2 y = \frac{x^3}{3} + c \qquad \Rightarrow \quad y = \frac{x}{3} + \frac{c}{x^2}$$

Examples of Integration by Parts

The method of integration by parts will be required in the next example of a first order linear ODE (Example 1.04.4). There are three main cases for integration by parts:

Example 1.04.2

Integrate $x^3 e^x$ with respect to *x*.

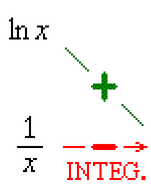


Therefore $\int x^3 e^x \, dx = e^x \left(x^3 - 3x^2 + 6x - 6 \right) + C$

This is an example where the table stops at a zero in the left column.

Example 1.04.3

Integrate $\ln x$ with respect to *x*.



Therefore $\int \ln x \, dx = x \ln x - \int \frac{x}{x} \, dx = x \ln x - \int 1 \, dx = x \ln x - x + C$

$\Rightarrow \int \ln x \, dx = x \left(\ln x - 1 \right) + C$

This is an example where the table stops at a row that can be integrated easily.

The third case, where the table stops at a row that is a multiple of the original integrand, follows in Example 1.04.4.

Example 1.04.4

An electrical circuit that contains a resistor, $R = 8\ \Omega$ (ohm), an inductor, $L = 0.02$ millihenry, and an applied emf, $E(t) = 2\cos(5t)$, is governed by the differential equation

$$L\frac{di}{dt} + Ri = \frac{dE}{dt}$$

Determine the current at any time $t \geq 0$, if initially there is a current of 1 ampere in the circuit.

First note that the inductance $L = 2\times10^{-5}$ H is very small. The ODE is therefore not very different from

$$0 + R\,i = dE/dt$$

which has the immediate solution

$$i = (1/R)\,dE/dt = (1/8)\times(-10\sin 5t)$$

We therefore anticipate that $i = -(5/4)\sin 5t$ will be a good approximation to the exact solution.

Substituting all values ($R = 8$, $L = 2\times10^{-5}$, $E = 2\cos 5t \Rightarrow E' = -10\sin 5t$) into the ODE yields

$$\frac{di}{dt} + 4\times10^5\, i = -5\times10^5 \sin 5t$$

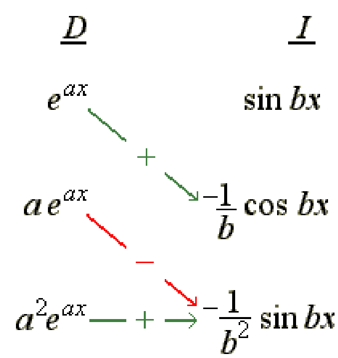
which is a linear first order ODE.

$P(t) = 400\,000$ and $R(t) = -500\,000\sin 5t \Rightarrow \quad h = \int P\,dt = 400\,000\,t$

$\Rightarrow$ integrating factor $= e^h = e^{400\,000t}$

$\Rightarrow \int e^h R\,dt = -500\,000\int e^{400\,000t}\sin 5t\,dt$

Integration by parts of the general case $\int e^{ax}\sin bx\,dx$:



$$\Rightarrow \int e^{ax}\sin bx\,dx = \left[-\frac{1}{b}e^{ax}\cos bx + \frac{a}{b^2}e^{ax}\sin bx\right] - \int \frac{a^2}{b^2}e^{ax}\sin bx\,dx$$

$$= \frac{1}{b^2}\left[e^{ax}\left(-b\cos bx + a\sin bx\right)\right] - \frac{a^2}{b^2}\int e^{ax}\sin bx\,dx$$

Example 1.04.4 (continued)

$$\Rightarrow \quad \left(1+\frac{a^2}{b^2}\right)\int e^{ax}\sin bx\,dx \;=\; \frac{1}{b^2}\left[e^{ax}\left(a\sin bx \;-\; b\cos bx\right)\right]$$

$$\Rightarrow \quad \int e^{ax}\sin bx\,dx \;=\; \frac{1}{a^2+b^2}\left[e^{ax}\left(a\sin bx \;-\; b\cos bx\right)\right] \;+\; C$$

Set $a = 400\,000$, $b = 5$ and $x = t$:

$$\Rightarrow \quad \int e^{h}R\,dt \;=\; -500\,000\,\frac{1}{400\,000^2+5^2}\,e^{400000t}\left(400\,000\sin 5t - 5\cos 5t\right)$$

The general solution is

$$i(t) \;=\; e^{-h}\left(\int e^{h}R\,dt \;+\; C\right)$$

$$\Rightarrow \quad i(t) \;=\; A\,e^{-400000t} \;-\; \frac{500\,000}{400\,000^2+25}\left(400\,000\sin 5t - 5\cos 5t\right)$$

But $i(0) = 1$

$$\Rightarrow \quad 1 \;=\; A \;-\; \frac{500\,000}{400\,000^2+25}\left(0 \;-\; 5\right)$$

$$\Rightarrow \quad A \;=\; (400\,000^2 + 25 - 2\,500\,000)\;/\;(400\,000^2 + 25)$$

Therefore the complete solution is [exactly]

$$\boxed{\;i(t) \;=\; \frac{159\,997\,500\,025\,e^{-400000\,t} \;-\; 500\,000\left(400\,000\sin 5t \;-\; 5\cos 5t\right)}{160\,000\,000\,025}\;}$$

To an excellent approximation, this complete solution is

$$\Rightarrow \quad i(t) \;\approx\; e^{-400000t} \;-\; \frac{5}{4}\sin 5t$$

After only a few microseconds, the transient term is negligible.
The complete solution is then, to an excellent approximation,

$$i(t) \;\approx\; -\;\frac{5}{4}\sin 5t$$

as before.

1.05 Bernoulli ODEs

The first order linear ODE is a special case of the Bernoulli ODE

$$\frac{dy}{dx} + P(x)\,y = R(x)\,y^n$$

If $n = 0$ then the ODE is linear.
If $n = 1$ then the ODE is separable.

For any other value of n, the change of variables $u = \dfrac{y^{1-n}}{1-n}$ will convert the Bernoulli ODE for y into a linear ODE for u.

$$\frac{du}{dx} = \frac{du}{dy}\frac{dy}{dx} = \frac{1-n}{1-n}y^{-n}\frac{dy}{dx} \quad \Rightarrow \quad \frac{dy}{dx} = y^n \frac{du}{dx}$$

The ODE transforms to

$$y^n \frac{du}{dx} + P(x)\,y = R(x)\,y^n \quad \Rightarrow \quad \frac{du}{dx} + P(x)\,y^{1-n} = R(x)$$

We therefore obtain the linear ODE for u:

$$\frac{du}{dx} + \big((1-n)P(x)\big)u = R(x)$$

whose solution is

$$\boxed{\frac{y^{1-n}}{1-n} = u(x) = e^{-h(x)}\left(\int e^{h(x)}R(x)\,dx + C\right), \quad \text{where} \quad h(x) = (1-n)\int P(x)\,dx}$$

together with the singular solution $y \equiv 0$ in the cases where $n > 0$.

Example 1.05.1

Find the general solution of the logistic population model

$$\frac{dy}{dx} = ay - by^2$$

where a, b are positive constants.

The Bernoulli equation is

$$\frac{dy}{dx} + (-a)y = (-b)y^2$$

with $P = -a, \quad R = -b, \quad n = 2.$

$$h = (1-n)\int P\,dx = (-1)\int -a\,dx = ax$$

Integrating factor $e^h = e^{ax}$

$$\int e^h R\,dx = \int e^{ax}(-b)\,dx = -\frac{b}{a}e^{ax} \qquad \text{(Note that } a > 0\text{)}$$

$$\frac{y^{-1}}{-1} = u = e^{-h}\left(\int e^h R\,dx + C\right) = e^{-ax}\left(-\frac{b}{a}e^{ax} + C\right)$$

$$\Rightarrow \quad y = \frac{a}{b - Ae^{-ax}}$$

Note that

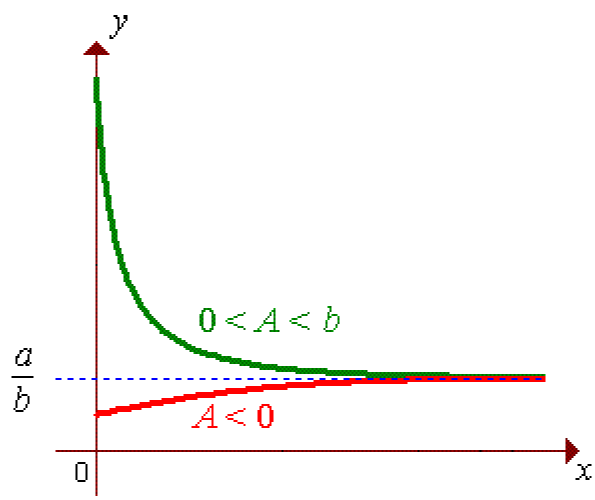
$$y(0) = \frac{a}{b-A} \quad \Rightarrow \quad A = b - \frac{a}{y(0)} \qquad \text{and} \qquad \lim_{x\to\infty} y = \frac{a}{b}$$

Also $y \equiv 0$ is a solution to the original ODE that is not included in the above solution for any finite value of the arbitrary constant A.

The general solution is

$$\boxed{y = \frac{a}{b - Ae^{-ax}} \quad \text{or} \quad y \equiv 0}$$

[Note that the initial condition is not positive and there is a discontinuity in y at $x = \frac{1}{a}\ln\frac{A}{b}$ if $A \geq b$ is true.]



1.06 Second Order Homogeneous Linear ODEs

The general second order linear ordinary differential equation with constant real coefficients may be written in the form

$$\frac{d^2 y}{dx^2} + p\frac{dy}{dx} + q\,y = r(x)$$

If, in addition, the right-side function $r(x)$ is identically zero, then the ODE is said to be **homogeneous**. Otherwise it is inhomogeneous.

The most general possible solution y_C to the homogeneous ODE $y'' + p\,y' + q\,y = 0$ is called the **complementary function**.
A solution y_P to the inhomogeneous ODE $y'' + p\,y' + q\,y = r(x)$ is called the **particular solution**.
The linearity of the ODE leads to the following two properties:
Any linear combination of two solutions to the homogeneous ODE is another solution to the homogeneous ODE; and
The sum of any solution to the homogeneous ODE and a particular solution is another solution to the inhomogeneous ODE.

It can be shown that the following is a valid method for obtaining the complementary function:

From the ODE $y'' + p\,y' + q\,y = r(x)$ form the **auxiliary equation** (or "characteristic equation")

$$\lambda^2 + p\lambda + q = 0$$

If the roots λ_1, λ_2 of this quadratic equation are distinct, then a basis for the entire set of possible complementary functions is $\{y_1, y_2\} = \left\{e^{\lambda_1 x}, e^{\lambda_2 x}\right\}$.

If the roots are not real (and therefore form a complex conjugate pair $a \pm bj$), then the basis can be expressed instead as the equivalent real set $\left\{e^{ax}\cos bx,\ e^{ax}\sin bx\right\}$.

If the roots are equal (and therefore real), then a basis for the entire set of possible complementary functions is $\{y_1, y_2\} = \left\{e^{\lambda x}, x e^{\lambda x}\right\}$.
The complementary function, in the form that captures all possibilities, is then

$$y_C = A y_1 + B y_2$$

where A and B are arbitrary constants.

Example 1.06.1

A simple unforced mass-spring system (with damping coefficient per unit mass = 6 s^{-1} and restoring coefficient per unit mass = 9 s^{-2}) is released from rest at an extension 1 m beyond its equilibrium position ($s = 0$). Find the position $s(t)$ at all subsequent times t.

The simple mass-spring system may be modelled by a second order linear ODE.

The $\dfrac{d^2s}{dt^2}$ term represents the acceleration of the mass, due to the net force.

The $\dfrac{ds}{dt}$ term represents the friction (damping) term.

The s term represents the restoring force.

The model is

$$\frac{d^2s}{dt^2} + 6\frac{ds}{dt} + 9s = 0$$

The auxiliary equation is

$$\lambda^2 + 6\lambda + 9 = 0 \quad \Rightarrow \quad (\lambda+3)^2 = 0 \quad \Rightarrow \quad \lambda = -3, -3$$

The roots are equal, so the basis functions for the complementary function are

$$\{s_1, s_2\} = \left\{e^{-3t}, t\,e^{-3t}\right\}$$

The ODE is homogeneous, so its **general solution** is also its complementary function:

$$s(t) = Ae^{-3t} + Bt\,e^{-3t} = (A + Bt)e^{-3t}$$

However, we have two additional items of information, (the **initial conditions**), which allow us to determine the values of the two arbitrary constants.

Initial displacement

$$s(0) = 1 \quad \Rightarrow \quad (A+0)e^0 = 1 \quad \Rightarrow \quad A = 1$$

$$s'(t) = (B - 3A - 3Bt)e^{-3t} = (B - 3 - 3Bt)e^{-3t}$$

Initial speed (released from rest)

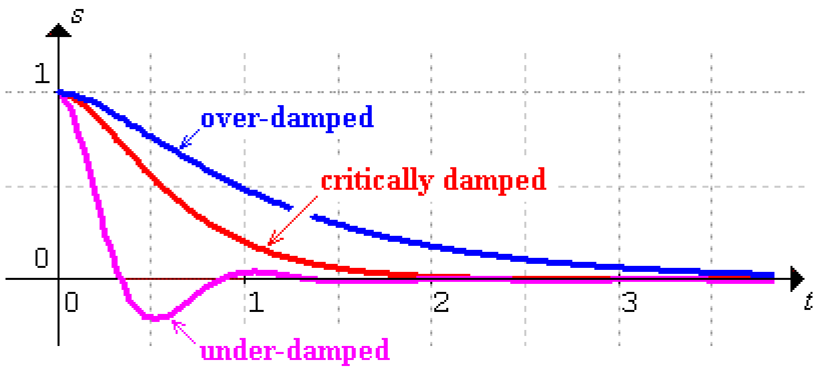
$$s'(0) = 0 \quad \Rightarrow \quad (B - 3 - 0)e^0 = 0 \quad \Rightarrow \quad B = 3$$

Therefore the complete solution is

$$\boxed{s(t) = (1+3t)e^{-3t}}$$

Example 1.06.1 (continued)

This is an example of **critical damping**.
Real distinct roots for λ correspond to **over-damping**.
Complex conjugate roots for λ correspond to **under-damping** (damped oscillations).



Illustrated here are a critically damped case $s(t) = (1+3t)e^{-3t}$ (the solution to Example 1.06.1), an over-damped case $s(t) = \frac{1}{3}\left(4e^{-t} - e^{-4t}\right)$ and an under-damped case $s(t) = e^{-3t}\left(\cos 6t + \frac{1}{2}\sin 6t\right)$, all of which share the same initial conditions $s(0) = 1$ and $s'(0) = 0$.

1.07 Variation of Parameters

A particular solution y_P to the inhomogeneous ODE $y'' + p\,y' + q\,y = r(x)$ may be constructed from the set of basis functions $\{y_1, y_2\}$ for the complementary function by varying the parameters:

Try $y_P(x) = u(x)\,y_1(x) + v(x)\,y_2(x)$, where the functions $u(x)$ and $v(x)$ are such that

(i) y_P is a solution of $y'' + p\,y' + q\,y = r(x)$ and

(ii) one free constraint is imposed, to ease the search for $u(x)$ and $v(x)$.

Substituting $y_P = u\,y_1 + v\,y_2$ into the ODE,

$$(u\,y_1 + v\,y_2)'' + p(u\,y_1 + v\,y_2)' + q(u\,y_1 + v\,y_2) = r$$

$$\Rightarrow \left((u'\,y_1 + v'\,y_2)' + (u\,y_1' + v\,y_2')'\right)$$

$$+ \; p\left((u'\,y_1 + v'\,y_2) + (u\,y_1' + v\,y_2')\right) + q(u\,y_1 + v\,y_2) = r$$

Imposing the free constraint $u'\,y_1 + v'\,y_2 \equiv 0$ simplifies the above expression to

$$(0 + u'\,y_1' + v'\,y_2' + u\,y_1'' + v\,y_2'') + p(0 + u\,y_1' + v\,y_2') + q(u\,y_1 + v\,y_2) = r$$

$$\Rightarrow u(y_1'' + p\,y_1' + q\,y_1) + v(y_2'' + p\,y_2' + q\,y_2) + u'\,y_1' + v'\,y_2' = r$$

But y_1 and y_2 are solutions of the homogeneous ODE $y'' + p\,y' + q\,y = 0$.

Therefore $0 + 0 + u'\,y_1' + v'\,y_2' = r$ is our other constraint.

Rewrite the two constraints together as a matrix equation:

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} 0 \\ r \end{bmatrix}$$

Using Cramer's rule to solve this matrix equation for u' and v' we obtain

$u' = \dfrac{W_1}{W}$ and $v' = \dfrac{W_2}{W}$, where $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ (the Wronskian),

and $W_1 = \begin{vmatrix} 0 & y_2 \\ r & y_2' \end{vmatrix} = -y_2 r$, $W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & r \end{vmatrix} = +y_1 r$

Integrate to find $u(x)$ and $v(x)$, then construct $y_P(x) = u(x)\,y_1(x) + v(x)\,y_2(x)$.

[space to continue the derivation of the method of variation of parameters]

Example 1.07.1

A mass spring system is at rest until the instant $t = 3$, when a sudden hammer blow, of impulse 10 Ns, sets the system into motion. No further external force is applied to the system, which has a mass of 1 kg, a restoring force coefficient of 26 kg s^{-2} and a friction coefficient of 2 kg s^{-1}. The response $x(t)$ at any time $t > 0$ is governed by the differential equation

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 26x = 10\,\delta(t-3)$$

(where $\delta(t-a)$ is the Dirac delta function),
together with the initial conditions $x(0) = x'(0) = 0$.
Find the complete solution to this initial value problem.

A.E.: $\lambda^2 + 2\lambda + 26 = 0 \quad \Rightarrow \quad \lambda = \dfrac{-2 \pm \sqrt{4 - 4\times 26}}{2} = -1 \pm 5j$

C.F.: $x_C = Ax_1 + Bx_2$, where $x_1 = e^{-t}\cos 5t$ and $x_2 = e^{-t}\sin 5t$

Define the abbreviations

$c = \cos 5t$, $s = \sin 5t$ $\Delta = \delta(t-3)$, and $E = e^{-t}$, then $r(t) = 10\,\Delta$.

P.S.: $r(t)$ is such that the method of undetermined coefficients cannot be used.

$$W = \begin{vmatrix} x_1 & x_2 \\ x_1' & x_2' \end{vmatrix} = \begin{vmatrix} Ec & Es \\ E(-5s-c) & E(5c-s) \end{vmatrix} = E^2\left(5c^2 - cs + 5s^2 + cs\right) = 5E^2$$

$$W_1 = \begin{vmatrix} 0 & x_2 \\ r & x_2' \end{vmatrix} = -x_2 r = -Es\cdot 10\Delta$$

$$u' = \frac{W_1}{W} = \frac{-Es\cdot 10\Delta}{5E^2} = -2E^{-1}s\,\Delta$$

$$\Rightarrow \quad u = -2\int e^{+t}\sin 5t\;\delta(t-3)\,dt = -2e^3(\sin 15)\,H(t-3)$$

(using the sifting property of the Dirac delta function in integrals,

$$\int_c^d f(t)\,\delta(t-a)\,dt = \begin{cases} f(a) & (\text{if} \quad c < a < d) \\ 0 & (a < c \ \text{ or } \ a > d) \end{cases}$$

and where $H(t-a) = \begin{cases} 0 & (t < a) \\ 1 & (t \ge a) \end{cases}$ is the Heaviside (unit step) function.)

$$W_2 = \begin{vmatrix} x_1 & 0 \\ x_1' & r \end{vmatrix} = +x_1 r = Ec\cdot 10\Delta$$

$$v' = \frac{W_2}{W} = \frac{Ec\cdot 10\Delta}{5E^2} = 2E^{-1}c\,\Delta$$

$$\Rightarrow \quad v = 2\int e^{+t}\cos 5t\;\delta(t-3)\,dt = 2e^3(\cos 15)\,H(t-3)$$

Example 1.07.1 (continued)

Using the trigonometric identity $\sin(A - B) = \sin A \cos B - \cos A \sin B$,

$$x_P = u\,x_1 + v\,x_2 = 2e^3 H(t-3)\,e^{-t}\left(-(\sin 15)\cos 5t + (\cos 15)\sin 5t\right)$$

$$= 2e^{-(t-3)} H(t-3)\sin(5t-15)$$

G.S. $x = x_C + x_P$:

$$x(t) = e^{-t}\left(A\cos 5t + B\sin 5t\right) + 2e^{-(t-3)}\sin\left(5(t-3)\right)H(t-3)$$

But, for $t < 3$, the system is undisturbed, at rest at equilibrium, so that

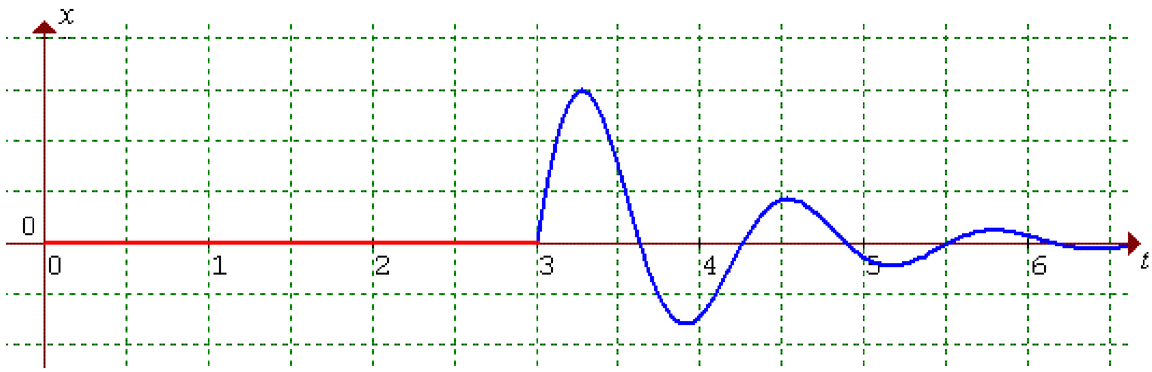
$$x(t) = e^{-t}\left(A\cos 5t + B\sin 5t\right) + 0\,; \qquad x(0) = x'(0) = r(t) = 0$$

$\Rightarrow \quad A = B = 0.$

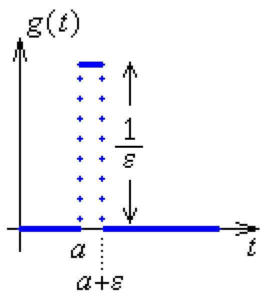
The complete solution is therefore

$$\boxed{x(t) = 2e^{-(t-3)}\sin 5(t-3)\,H(t-3)}$$

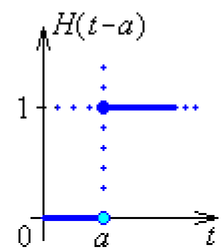
This complete solution is continuous at $t = 3$.
It is not differentiable at $t = 3$, because of the infinite discontinuity of the Dirac delta function inside $r(t)$ at $t = 3$.



Note: $\delta(t-a) = \lim_{\varepsilon \to 0} g(t; a, \varepsilon)$ $\qquad H(t-a) = \begin{cases} 0 & (t < a) \\ 1 & (t \geq a) \end{cases}$



[Total area = 1]



Example 1.07.2

Find the general solution of the ODE $y'' + 2y' - 3y = x^2 + e^{2x}$.

A.E.: $\lambda^2 + 2\lambda - 3 = 0$

$\Rightarrow \quad (\lambda + 3)(\lambda - 1) = 0 \quad \Rightarrow \quad \lambda = -3, 1$

$y_1 = e^{-3x}, \quad y_2 = e^x, \quad r = x^2 + e^{2x}$

Particular Solution by Variation of Parameters:

$$W(x) = \det\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} = \det\begin{bmatrix} e^{-3x} & e^x \\ -3e^{-3x} & e^x \end{bmatrix} = 4e^{-2x}$$

$$W_1 = \det\begin{bmatrix} 0 & y_2 \\ r & y_2' \end{bmatrix} = -y_2 r = -e^x\left(x^2 + e^{2x}\right)$$

$$\Rightarrow \quad u' = \frac{W_1}{W} = \frac{-\left(x^2e^x + e^{3x}\right)}{4e^{-2x}} = -\left(\frac{x^2e^{3x} + e^{5x}}{4}\right)$$

$$\Rightarrow \quad u = -\frac{1}{4}\int\left(x^2e^{3x} + e^{5x}\right)dx$$

D		**I**
x^2		e^{3x}
	+	
$2x$		$\frac{1}{3}e^{3x}$
	−	
2		$\frac{1}{9}e^{3x}$
	+	
0		$\frac{1}{27}e^{3x}$

$$\Rightarrow \quad u = -\frac{1}{4}\left(\frac{e^{3x}}{27}\left(9x^2 - 6x + 2\right) + \frac{1}{5}e^{5x}\right)$$

$$W_2 = \det\begin{bmatrix} y_1 & 0 \\ y_1' & r \end{bmatrix} = +y_1 r = e^{-3x}\left(x^2 + e^{2x}\right)$$

Example 1.07.2 (continued)

$$\Rightarrow \quad v' = \frac{W_2}{W} = \frac{x^2 e^{-3x} + e^{-x}}{4e^{-2x}} = \frac{x^2 e^{-x} + e^x}{4}$$

$$\Rightarrow \quad v = \frac{1}{4}\int \left(x^2 e^{-x} + e^x\right) dx$$

D		**I**
x^2		e^{-x}
	$+$	
$2x$		$-e^{-x}$
	$-$	
2		$+e^{-x}$
	$+$	
0		$-e^{-x}$

$$\Rightarrow \quad v = \frac{1}{4}\left(e^{-x}\left(-x^2 - 2x - 2\right) + e^x\right)$$

$$y_P = u \cdot y_1 + v \cdot y_2 =$$

$$\frac{1}{4}\left\{\left(-\frac{e^{3x}}{27}\left(9x^2 - 6x + 2\right) - \frac{1}{5}e^{5x}\right)e^{-3x} + \frac{27}{27}\left(-e^{-x}\left(x^2 + 2x + 2\right) + e^x\right)e^x\right\}$$

$$= \frac{1}{4}\left\{\frac{1}{27}\left(-9x^2 + 6x - 2 - 27x^2 - 54x - 54\right) + \left(-\frac{1}{5} + 1\right)e^{2x}\right\}$$

$$= \frac{1}{4}\left\{\frac{1}{27}\left(-36x^2 - 48x - 56\right) + \frac{4}{5}e^{2x}\right\}$$

Therefore

$$y_P = \frac{1}{5}e^{2x} - \frac{1}{27}\left(9x^2 + 12x + 14\right)$$

and the general solution is

$$y(x) = \underline{\underline{A\,e^{-3x} + B\,e^x + \frac{1}{5}e^{2x} - \frac{1}{27}\left(9x^2 + 12x + 14\right)}}$$

1.08 Method of Undetermined Coefficients

When trying to find the particular solution of the inhomogeneous ODE

$$\frac{d^2 y}{dx^2} + p\frac{dy}{dx} + q\,y = r(x)$$

an alternative method to variation of parameters is available only when $r(x)$ is one of the following special types:

e^{kx}, $\cos kx$, $\sin kx$, $\sum_{k=1}^{n} a_k x^k$ and any linear combinations of these types and any products of these types. When it is available, this method is often faster than the method of variation of parameters.

The method involves the substitution of a form for y_P that resembles $r(x)$, with coefficients yet to be determined, into the ODE.

If $r(x) = c\,e^{kx}$, then try $y_P = d\,e^{kx}$, with the coefficient *d* to be determined.

If $r(x) = a\cos kx$ or $b\sin kx$, then try $y_P = c\cos kx + d\sin kx$, with the coefficients *c* and *d* to be determined.

If $r(x)$ is an n^{th} order polynomial function of *x*, then set y_P equal to an n^{th} order polynomial function of *x*, with all $(n + 1)$ coefficients to be determined.

However, if $r(x)$ contains a constant multiple of either part of the complementary function $(y_1 \text{ or } y_2)$, then that part must be multiplied by *x* in the trial function for y_P.

Example 1.08.1 (Example 1.07.2 again)

Find the general solution of the ODE $y'' + 2y' - 3y = x^2 + e^{2x}$.

A.E.: $\lambda^2 + 2\lambda - 3 = 0$

$\Rightarrow \quad (\lambda + 3)(\lambda - 1) = 0 \quad \Rightarrow \quad \lambda = -3, 1$

C.F.: $y_C = A e^{-3x} + B e^x$

Particular Solution by Undetermined Coefficients:

$r(x) = x^2 + e^{2x}$, so try $y_P = ax^2 + bx + c + d e^{2x}$

Then $y'' + 2y' - 3y =$

$$\begin{array}{rrrrrrrrrcl}
 & & & & & 2a & + & 4d\,e^{2x} & \leftarrow & y_P'' \\
+ & & & 4a\,x & + & 2b & + & 4d\,e^{2x} & \leftarrow & +2y_P' \\
+ & -3a\,x^2 & - & 3b\,x & - & 3c & - & 3d\,e^{2x} & \leftarrow & -3y_P \\
\hline
= & 1\,x^2 & + & 0\,x & + & 0 & + & 1\,e^{2x} & \leftarrow & = r
\end{array}$$

Matching coefficients:

$x^2: \quad -3a = 1 \quad \Rightarrow \quad a = -\tfrac{1}{3}$

$x^1: \quad 4\left(-\tfrac{1}{3}\right) - 3b = 0 \quad \Rightarrow \quad b = -\tfrac{4}{9}$

$x^0: \quad 2\left(-\tfrac{1}{3}\right) + 2\left(-\tfrac{4}{9}\right) - 3c = 0 \quad \Rightarrow \quad c = -\frac{2}{3}\left(\frac{3+4}{9}\right) = -\tfrac{14}{27}$

$e^{2x}: \quad (4 + 4 - 3)d = 1 \quad \Rightarrow \quad d = \tfrac{1}{5}$

G.S.: $y(x) = y_C(x) + y_P(x)$

Therefore

$$y(x) = \underline{\underline{A e^{-3x} + B e^x + \tfrac{1}{5} e^{2x} - \tfrac{1}{27}\left(9x^2 + 12x + 14\right)}}$$

Example 1.08.2

Find the general solution of the ODE

$$\frac{d^2 y}{dx^2} + 4\frac{dy}{dx} + 4y = e^{-2x}$$

A.E.: $\lambda^2 + 4\lambda + 4 = 0 \quad \Rightarrow \quad (\lambda+2)^2 = 0 \quad \Rightarrow \quad \lambda = -2, -2$

C.F.: $y_C = (Ax+B)e^{-2x}$

P.S.:

$r(x) = e^{-2x}$, but both e^{-2x} and $x\,e^{-2x}$ are in the C.F.

Therefore try $y_P = c\,x^2 e^{-2x}$.

$$y_P'' + 4y_P' + 4y_P = c\Big(\big(4x^2-8x+2\big) + 4\big(-2x^2+2x\big) + 4\big(x^2\big)\Big)e^{-2x} = e^{-2x}$$

$$\Rightarrow \quad c\Big((4-8+4)x^2 + (-8+8)x + 2\Big) = 1$$

$$\Rightarrow \quad c = \tfrac{1}{2}$$

Therefore the general solution is

$$\boxed{y(x) = \left(\tfrac{1}{2}x^2 + Ax + B\right)e^{-2x}}$$

Again, this is much faster than variation of parameters.
However, the method of variation of parameters may be employed regardless of the form of the right side $r(x)$, while the method of undetermined coefficients may be used only for a narrow range of forms of $r(x)$.

1.09 Laplace Transforms

Laplace transforms can convert some initial value problems into algebra problems. It is assumed here that students have met Laplace transforms before. Only the key results are displayed here, before they are employed to solve some initial value problems.

The Laplace transform of a function $f(t)$ is the integral

$$\boxed{F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt}$$

where the integral exists.

Some standard transforms and properties are:

Linearity:

$$\mathcal{L}\{a f(t) + b g(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\} \qquad (a, b = \text{constants})$$

Polynomial functions:

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}} \quad \Rightarrow \quad \mathcal{L}^{-1}\left\{\frac{1}{s^n}\right\} = \frac{t^{n-1}}{(n-1)!}$$

First Shift Theorem:

$$\mathcal{L}\{f(t)\} = F(s) \quad \Rightarrow \quad \mathcal{L}\{e^{at} f(t)\} = F(s-a)$$

$$\Rightarrow \quad \mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at} \quad \text{and} \quad \mathcal{L}^{-1}\left\{\frac{1}{(s-a)^n}\right\} = \frac{t^{n-1}e^{at}}{(n-1)!}$$

Trigonometric Functions:

$$\mathcal{L}\{e^{at}\sin\omega t\} = \frac{\omega}{(s-a)^2 + \omega^2} \quad \Rightarrow \quad \mathcal{L}^{-1}\left\{\frac{1}{(s-a)^2 + \omega^2}\right\} = \frac{e^{at}\sin\omega t}{\omega}$$

$$\mathcal{L}\{e^{at}\cos\omega t\} = \frac{s-a}{(s-a)^2 + \omega^2} \quad \Rightarrow \quad \mathcal{L}^{-1}\left\{\frac{s-a}{(s-a)^2 + \omega^2}\right\} = e^{at}\cos\omega t$$

Derivatives:

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$$

$$\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - s f(0) - f'(0)$$

Integration:

$$\mathcal{L}^{-1}\left\{\frac{1}{s}G(s)\right\} = \int_0^t \mathcal{L}^{-1}\{G(s)\}\,d\tau$$

$$\frac{d}{ds}\mathcal{L}\{f(t)\} = -\mathcal{L}\{t\,f(t)\} \quad\Rightarrow\quad \mathcal{L}^{-1}\{F'(s)\} = -t\cdot\mathcal{L}^{-1}\{F(s)\}$$

Second shift theorem:

$$\mathcal{L}^{-1}\{F(s)\} = f(t) \quad\Rightarrow\quad \mathcal{L}^{-1}\{e^{-as}F(s)\} = H(t-a)\,f(t-a)$$

where $H(t-a) = \begin{cases} 0 & (t<a) \\ 1 & (t\ge a) \end{cases}$ is the Heaviside (unit step) function.

Dirac delta function

$$\mathcal{L}\{\delta(t-a)\} = e^{-as}$$

where $\int_c^d f(t)\delta(t-a)\,dt = \begin{cases} f(a) & (\text{if} \quad c<a<d) \\ 0 & (a<c \ \text{ or } \ a>d) \end{cases}$.

For a periodic function $f(t)$ with fundamental period p,

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-ps}}\int_0^p e^{-st} f(t)\,dt$$

Convolution:

$$\mathcal{L}^{-1}\{F(s)G(s)\} = \mathcal{L}^{-1}\{F(s)\} * \mathcal{L}^{-1}\{G(s)\}$$

where $(f * g)(t)$ denotes the convolution of $f(t)$ and $g(t)$ and is defined by

$$(f*g)(t) = \int_0^t f(\tau)\,g(t-\tau)\,d\tau$$

The identity function for convolution is the Dirac delta function:

$$\delta(t-a) * f(t) = f(t-a)H(t-a) \quad\Rightarrow\quad \delta(t) * f(t) = f(t)$$

Here is a summary of inverse Laplace transforms.

$F(s)$	$f(t)$
$\int_0^\infty e^{-st} f(t)\, dt$	$f(t)$
$\dfrac{1}{s^n} \quad (n \in \mathbb{N})$	$\dfrac{t^{n-1}}{(n-1)!}$
$\dfrac{1}{\sqrt{s}}$	$\dfrac{1}{\sqrt{\pi t}}$
$\dfrac{1}{s-a}$	e^{at}
$\dfrac{1}{(s-a)^n} \quad (n \in \mathbb{N})$	$\dfrac{t^{n-1} e^{at}}{(n-1)!}$
e^{-as}	$\delta(t-a)$
$\dfrac{e^{-as}}{s}$	$H(t-a)$
$\dfrac{1}{s^2+\omega^2}$	$\dfrac{\sin \omega t}{\omega}$
$\dfrac{1}{(s-a)^2+\omega^2}$	$\dfrac{e^{at} \sin \omega t}{\omega}$
$\dfrac{1}{(s-a)^2-b^2}$	$\dfrac{e^{at} \sinh bt}{b}$
$\dfrac{(s-a)}{(s-a)^2+\omega^2}$	$e^{at} \cos \omega t$
$\dfrac{(s-a)}{(s-a)^2-b^2}$	$e^{at} \cosh bt$

$F(s)$	$f(t)$
$\dfrac{1}{s(s^2+\omega^2)}$	$\dfrac{1-\cos \omega t}{\omega^2}$
$\dfrac{1}{s^2(s^2+\omega^2)}$	$\dfrac{\omega t - \sin \omega t}{\omega^3}$
$\dfrac{1}{(s^2+\omega^2)^2}$	$\dfrac{\sin \omega t - \omega t \cos \omega t}{2\omega^3}$
$\dfrac{s}{(s^2+\omega^2)^2}$	$\dfrac{t \sin \omega t}{2\omega}$
$\dfrac{s^2-\omega^2}{(s^2+\omega^2)^2}$	$t \cos \omega t$
$\dfrac{1}{s} \tanh\left(\dfrac{as}{2}\right)$	Square wave, period $2a$, amplitude 1
$\dfrac{1}{s^2} \tanh\left(\dfrac{as}{2}\right)$	Triangular wave, period $2a$, amplitude a
$\dfrac{b}{as^2} - \dfrac{b}{s(e^{as}-1)}$	Sawtooth wave, period a, amplitude b
$\{ s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - s^{n-3} f''(0) - \ldots - s f^{(n-2)}(0) - f^{(n-1)}(0) \}$	$\dfrac{d^n f}{dt^n}$
$\dfrac{1}{s} F(s)$	$\int_0^t f(\tau)\, d\tau$
$\dfrac{dF}{ds}$	$-t\, f(t)$

<u>Example 1.09.1</u> (Example 1.08.2 again)

Find the general solution of the ODE

$$\frac{d^2 y}{dx^2} + 4\frac{dy}{dx} + 4y = e^{-2x}$$

The initial conditions are unknown, so let $a = y(0)$ and $b = y'(0)$.
Taking the Laplace transform of the initial value problem,

$$\left(s^2 Y - s\cdot a - b\right) + 4\left(sY - a\right) + 4Y = \frac{1}{s+2}$$

$$\Rightarrow \left(s^2+4s+4\right)Y = as+4a+b + \frac{1}{s+2}$$

$$\Rightarrow Y = \frac{as+4a+b}{\left(s+2\right)^2} + \frac{1}{\left(s+2\right)^3} = a\frac{\left(s+2\right)}{\left(s+2\right)^2} + \frac{2a+b}{\left(s+2\right)^2} + \frac{1}{\left(s+2\right)^3}$$

Note that $\mathscr{L}^{-1}\left\{\frac{1}{\left(s+a\right)^n}\right\} = \frac{x^{n-1}e^{-ax}}{\left(n-1\right)!}$

$$\Rightarrow y = a\,\mathscr{L}^{-1}\left\{\frac{1}{s+2}\right\} + \left(2a+b\right)\mathscr{L}^{-1}\left\{\frac{1}{\left(s+2\right)^2}\right\} + \mathscr{L}^{-1}\left\{\frac{1}{\left(s+2\right)^3}\right\}$$

$$\Rightarrow y = a\,e^{-2x} + \left(2a+b\right)x\,e^{-2x} + \frac{x^2 e^{-2x}}{2!}$$

Introducing the new arbitrary constants $A = 2a + b$ and $B = a$, we recover the general solution

$$\boxed{y = \left(\tfrac{1}{2}x^2 + Ax + B\right)e^{-2x}}$$

Example 1.09.2 (Example 1.07.1 again)

Find the complete solution to the initial value problem

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 26x = 10\,\delta(t-3)$$

(where $\delta(t-a)$ is the Dirac delta function),
together with the initial conditions $x(0) = x'(0) = 0$.

Let $X(s) = \mathcal{L}\{x(t)\}$ be the Laplace transform of the solution $x(t)$.
Taking the Laplace transform of the initial value problem,

$$\left(s^2 X - 0 - 0\right) + 2\left(sX - 0\right) + 26X = 10e^{-3s}$$

$$\Rightarrow \quad \left(s^2+2s+26\right)X = 10e^{-3s} \qquad \Rightarrow \quad X = \frac{10e^{-3s}}{(s+1)^2+5^2} = \mathcal{L}\left\{2e^{-t}\sin 5t\right\}e^{-3s}$$

By the second shift theorem, it then follows that the complete solution is

$$x(t) = 2e^{-t}\sin 5t\Big|_{t\to t-3} H(t-3) = 2e^{-(t-3)}\sin 5(t-3)\,H(t-3)$$

This is a considerably faster solution than that provided by the method of variation of parameters (Example 1.07.1).

OR

$$X(s) = \frac{10e^{-3s}}{(s+1)^2+5^2} = \mathcal{L}\left\{2e^{-t}\sin 5t\right\}\cdot\mathcal{L}\left\{\delta(t-3)\right\} = \mathcal{L}\left\{2e^{-t}\sin 5t * \delta(t-3)\right\}$$

Using the convolution properties of the Dirac delta function,

$$x(t) = \left(2e^{-t}\sin 5t\right) * \delta(t-3) = 2e^{-(t-3)}\sin 5(t-3)\,H(t-3)$$

1.10 Series Solutions of ODEs

If the functions $p(x)$, $q(x)$ and $r(x)$ in the ODE

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = r(x)$$

are all analytic in some interval $x_o - h < x < x_o + h$ (and therefore possess Taylor series expansions around x_o with radii of convergence of at least h), then a series solution to the ODE around x_o with a radius of convergence of at least h exists:

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_o)^n, \qquad a_n = \frac{y^{(n)}(x_o)}{n!}$$

Example 1.10.1

Find a series solution as far as the term in x^3, to the initial value problem

$$\frac{d^2y}{dx^2} - x\frac{dy}{dx} + e^x y = 4; \qquad y(0) = 1, \quad y'(0) = 4$$

None of our previous methods apply to this problem.
The functions $-x$, e^x and 4 are all analytic everywhere.

The solution of this ODE, expressed as a power series, is

$$y(x) = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \ldots$$

But $y(0) = 1$ and $y'(0) = 4$.
From the ODE,

$$y'' = x y' - e^x y + 4 \quad \Rightarrow \quad y''(0) = 0 - y(0) + 4 = -1 + 4 = 3$$

Differentiating the ODE,

$$y''' = y' + x y'' - e^x y - e^x y' + 0 \quad \Rightarrow \quad y'''(0) = y'(0) + 0 - y(0) - y'(0) = -1$$

Therefore the first four terms of the solution are

$$\boxed{y(x) = 1 + 4x + \frac{3}{2}x^2 - \frac{1}{6}x^3 + \ldots}$$

Example 1.10.2

Find the general solution (as a power series about $x = 0$) to the ordinary differential equation

$$\frac{d^2y}{dx^2} + x^2 y = 0$$

Let the general solution be $y(x) = \sum_{n=0}^{\infty} a_n x^n$.

Then $y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$ and $y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$.

Substitute into the ODE:

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + x^2 \sum_{n=0}^{\infty} a_n x^n = 0 \Rightarrow \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^{n+2} = 0$$

Shift the indices on each summation so that the exponent of x is n in both cases:

$$\Rightarrow \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=2}^{\infty} a_{n-2} x^n = 0$$

Bring the two summations together for all terms from $n = 2$ onwards:

$$\Rightarrow 2\times 1 a_2 x^0 + 3\times 2 a_3 x^1 + \sum_{n=2}^{\infty} \left((n+2)(n+1) a_{n+2} + a_{n-2} \right) x^n = 0$$

But this equation must be true regardless of the choice of x.

Therefore the coefficient of each power of x must be zero. $\Rightarrow a_2 = a_3 = 0$

and $a_{n+2} = \dfrac{-a_{n-2}}{(n+2)(n+1)} \quad (n = 2, 3, 4, \ldots) \Rightarrow a_n = \dfrac{-a_{n-4}}{n(n-1)} \quad (n = 4, 5, 6, \ldots)$

$$\Rightarrow a_4 = \frac{-a_0}{4\times 3} = -\frac{a_0}{12}, \qquad a_5 = \frac{-a_1}{5\times 4} = -\frac{a_1}{20}$$

$$a_6 = \frac{-a_2}{6\times 5} = 0, \qquad a_7 = \frac{-a_3}{7\times 6} = 0$$

$$a_8 = \frac{-a_4}{8\times 7} = \frac{+a_0}{56\times 12} = \frac{+a_0}{672}, \qquad a_9 = \frac{-a_5}{9\times 8} = \frac{+a_1}{72\times 20} = \frac{+a_1}{1440}, \quad \ldots$$

and $a_0 = y(0)$ and $a_1 = y'(0)$ are arbitrary.

Therefore the general solution is

$$y(x) = a_0 + a_1 x + 0x^2 + 0x^3 - \frac{a_0}{12} x^4 - \frac{a_1}{20} x^5 + 0x^6 + 0x^7$$

$$+ \frac{a_0}{672} x^8 + \frac{a_1}{1440} x^9 + 0x^{10} + 0x^{11} + \ldots$$

or

$$\boxed{y(x) = a_0 \left(1 - \tfrac{1}{12} x^4 + \tfrac{1}{672} x^8 - \ldots\right) + a_1 \left(x - \tfrac{1}{20} x^5 + \tfrac{1}{1440} x^9 - \ldots\right)}$$

1.11 The Gamma Function

The gamma function $\Gamma(x)$ is a special function that will be needed in the solution of Bessel's ODE. $\Gamma(x)$ is a generalisation of the factorial function $n!$ from positive integers to most real numbers. For any positive integer n,
$n! = n \times (n-1) \times (n-2) \times \ldots \times 3 \times 2 \times 1$ (with $0!$ defined to be 1)

When x is a positive integer n, $\boxed{\Gamma(n) = (n-1)!}$

We know that $n! = n \times (n-1)!$
The gamma function has a similar recurrence relationship: $\Gamma(x+1) = x \cdot \Gamma(x)$

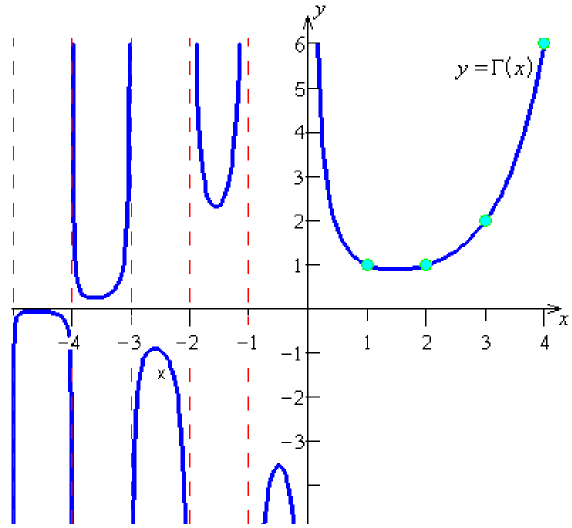
This allows $\Gamma(x)$ to be defined for non-integer negative x, using $\boxed{\Gamma(x) = \dfrac{\Gamma(x+1)}{x}}$

For example,
it can be shown that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

$$\Rightarrow \quad \Gamma\left(-\tfrac{1}{2}\right) = \frac{\Gamma\left(+\frac{1}{2}\right)}{-\frac{1}{2}} = -2\sqrt{\pi} \qquad \Rightarrow \quad \Gamma\left(-\tfrac{3}{2}\right) = \frac{\Gamma\left(-\frac{1}{2}\right)}{-\frac{3}{2}} = +\frac{4\sqrt{\pi}}{3}, \text{ etc.}$$

$\Gamma(x)$ is infinite when x is a negative integer or zero. It is well defined for all other real numbers x.

In this graph of $y = \Gamma(x)$, values of the factorial function (at positive integer values of x) are highlighted.



There are several ways to define the gamma function, such as

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt \qquad (x > 0)$$

and

$$\Gamma(x) = \lim_{n \to \infty} \frac{n!\, n^x}{x(x+1)\ldots(x+n)}$$

A related special function is the **beta function**:

$$B(m,n) = \int_0^1 t^{m-1}(1-t)^{n-1}\,dt = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

Among the many results involving the gamma function are:

For the closed region V in the first octant, bounded by the coordinate planes and the surface $\left(\frac{x}{a}\right)^\alpha + \left(\frac{y}{b}\right)^\beta + \left(\frac{z}{c}\right)^\gamma = 1$, with all constants positive,

$$I = \iiint_V x^{p-1} y^{q-1} z^{r-1} dx\,dy\,dz = \frac{a^p b^q c^r}{\alpha\beta\gamma} \cdot \frac{\Gamma\left(\frac{p}{\alpha}\right)\Gamma\left(\frac{q}{\beta}\right)\Gamma\left(\frac{r}{\gamma}\right)}{\Gamma\left(1+\frac{p}{\alpha}+\frac{q}{\beta}+\frac{r}{\gamma}\right)}$$

For the closed area A in the first quadrant, bounded by the coordinate axes and the curve $\left(\frac{x}{a}\right)^\alpha + \left(\frac{y}{b}\right)^\beta = 1$, with all constants positive,

$$I = \iint_A x^{p-1} y^{q-1} dx\,dy = \frac{a^p b^q}{\alpha\beta} \cdot \frac{\Gamma\left(\frac{p}{\alpha}\right)\Gamma\left(\frac{q}{\beta}\right)}{\Gamma\left(1+\frac{p}{\alpha}+\frac{q}{\beta}\right)}$$

Example 1.11.1

Establish the formula for the area enclosed by an ellipse.

The Cartesian equation of a standard ellipse is $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$.

Set $\alpha = \beta = 2$ and $p = q = 1$, then

$$A = 4I = 4\iint_A 1\,dx\,dy = 4\frac{a^1 b^1}{2\times 2} \cdot \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(1+\frac{1}{2}+\frac{1}{2}\right)} = ab \cdot \frac{\left(\Gamma\left(\frac{1}{2}\right)\right)^2}{\Gamma(2)} = ab\frac{\left(\sqrt{\pi}\right)^2}{1!} = \pi ab$$

1.12 Bessel and Legendre ODEs

Frobenius Series Solution of an ODE

If the ODE

$$P(x)\,y'' + Q(x)\,y' + R(x)\,y = F(x)$$

is such that $P(x_o) = 0$, but $(x-x_o)\dfrac{Q(x)}{P(x)}$, $(x-x_o)^2\dfrac{R(x)}{P(x)}$ and $\dfrac{F(x)}{P(x)}$ are all analytic at x_o, then $x = x_o$ is a **regular singular point** of the ODE.

A Frobenius series solution of the ODE about $x = x_o$ exists:

$$y(x) = \sum_{n=0}^{\infty} c_n (x-x_o)^{n+r}$$

for some real number(s) r and for some set of values $\{ c_n \}$.

Example 1.12.1

Find a solution of Bessel's ordinary differential equation of order ν, ($\nu \geq 0$),

$$x^2 y'' + x\,y' + \left(x^2 - \nu^2\right) y = 0$$

$P(0) = 0 \Rightarrow x_o = 0$ is a singular point.

$$(x-x_o)\frac{Q(x)}{P(x)} = x\frac{x}{x^2} = 1 \quad , \quad (x-x_o)^2\frac{R(x)}{P(x)} = x^2\frac{\left(x^2-\nu^2\right)}{x^2} = x^2 - \nu^2$$

and $\dfrac{F(x)}{P(x)} = 0$

Therefore $x_o = 0$ is a regular singular point of Bessel's equation.

Substitute the Frobenius series $y(x) = \sum_{n=0}^{\infty} c_n x^{n+r}$ into the ODE:

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-2+2} + \sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1+1}$$

$$+ \sum_{n=0}^{\infty} c_n x^{n+r+2} - \sum_{n=0}^{\infty} \nu^2 c_n x^{n+r} = 0$$

Example 1.12.1 (continued)

Adjust the index on the third summation so that the exponents of x match:

$$\sum_{n=0}^{\infty} c_n \left[(n+r)(n+r-1+1) - \nu^2 \right] x^{n+r} + \sum_{n=2}^{\infty} c_{n-2} x^{n+r} = 0$$

The summations can be combined for $n = 2$ onwards:

$$\left(r^2 - \nu^2\right) c_0 x^r + \left[(r+1)^2 - \nu^2 \right] c_1 x^{r+1} + \sum_{n=2}^{\infty} \left[\left((n+r)^2 - \nu^2 \right) c_n + c_{n-2} \right] x^{n+r} = 0$$

Setting the coefficient of x^r (the lowest exponent present) to zero generates the **indicial equation** $r^2 - \nu^2 = 0 \quad \Rightarrow \quad r = \pm \nu$.

Examining the positive root, the series now becomes

$$0 + \left[(\nu+1)^2 - \nu^2 \right] c_1 x^{r+1} + \sum_{n=2}^{\infty} \left[\left((n+\nu)^2 - \nu^2 \right) c_n + c_{n-2} \right] x^{n+r} = 0$$

$$\Rightarrow \quad (2\nu+1) c_1 x^{r+1} + \sum_{n=2}^{\infty} \left[\left(2n\nu + n^2 \right) c_n + c_{n-2} \right] x^{n+r} = 0$$

But $\nu \ge 0 \quad \Rightarrow \quad 2\nu + 1 \ne 0 \quad \Rightarrow \quad c_1 = 0$

For $n \ge 2$, $\quad c_n = \dfrac{-c_{n-2}}{n(n+2\nu)}$

It then follows that this series must be even: $0 = c_1 = c_3 = c_5 = \ldots$ or $c_{2k-1} = 0 \quad \forall k \in \mathbb{N}$

For the even order terms, replace the index n by the even index $2k$ (where k is any natural number) and pursue the recurrence relation down to c_0:

$$c_{2k} = \frac{-c_{2k-2}}{2k(2k+2\nu)} = \frac{(-1)}{2^2 k(k+\nu)} c_{2(k-1)} = \frac{(-1)}{2^2 k(k+\nu)} \cdot \frac{(-1)}{2^2 (k-1)(k-1+\nu)} c_{2(k-2)}$$

$$= \frac{(-1)}{2^2 k(k+\nu)} \cdot \frac{(-1)}{2^2 (k-1)(k-1+\nu)} \cdot \frac{(-1)}{2^2 (k-2)(k-2+\nu)} c_{2(k-3)} = \cdots$$

$$= \frac{(-1)^k}{2^{2k} k(k-1)(k-2)\ldots(k-[k-1]) \cdot (k+\nu)(k-1+\nu)(k-2+\nu)\ldots(k-[k-1]+\nu)} c_{2(k-k)}$$

$$\Rightarrow \quad c_{2k} = \frac{(-1)^k}{2^{2k} k!(\nu+k)(\nu+k-1)\ldots(\nu+1)} c_0$$

$$\Rightarrow \quad c_{2k} = \frac{(-1)^k}{2^{2k} k!(\nu+k)(\nu+k-1)\ldots(\nu+1)} \cdot \frac{\Gamma(\nu+1)}{\Gamma(\nu+1)} c_0$$

$$\Rightarrow \quad c_{2k} = \frac{(-1)^k}{2^{2k} k!\,\Gamma(\nu+k+1)} \cdot \Gamma(\nu+1) c_0 \quad \left(= \frac{(-1)^k}{2^{2k} k!(\nu+k)!} \cdot \nu! c_0 \text{ if } \nu = 0,1,2,\ldots \right)$$

Example 1.12.1 (continued)

One Frobenius solution of Bessel's equation of order ν is therefore

$$\boxed{y(x) = c_0 \Gamma(\nu+1) \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k} k! \Gamma(\nu+k+1)} x^{2k+\nu} = c_0 \Gamma(\nu+1) J_\nu(x)}$$

where $J_\nu(x)$ is the **Bessel function of the first kind of order** $\boldsymbol{\nu}$.

It turns out that the Frobenius series found by setting $r = -\nu$ generates a second linearly independent solution $J_{-\nu}(x)$ of the Bessel equation only if ν is not an integer.

The Bessel ODE in standard form,

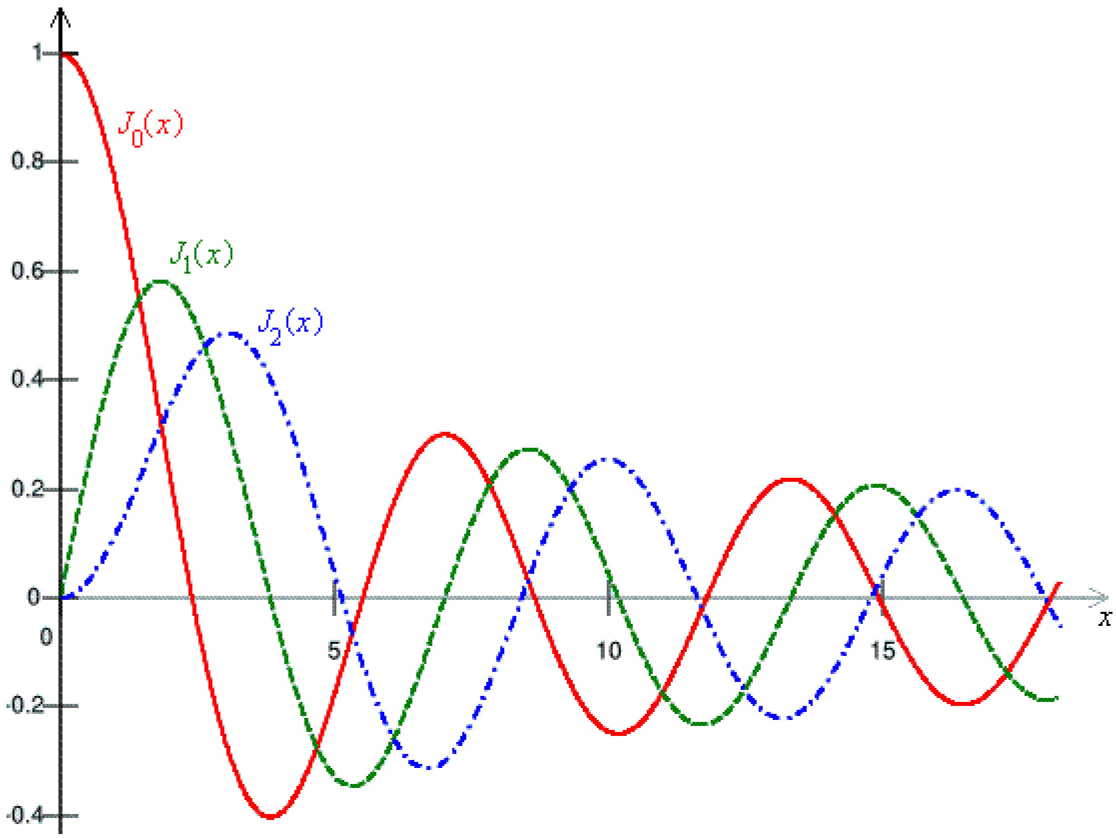
$$x^2 y'' + x y' + \left(x^2 - \nu^2\right) y = 0$$

has the general solution

$$y(x) = A J_\nu(x) + B Y_\nu(x)$$

unless ν is not an integer, in which case $Y_\nu(x)$ can be replaced by $J_{-\nu}(x)$.
$Y_\nu(x)$ is the Bessel function of the second kind.
When ν is an integer, $J_{-\nu}(x) = (-1)^\nu J_\nu(x)$.

Graphs of Bessel functions of the first kind, for $\nu = 0, 1, 2$:



The series expression for the Bessel function of the first kind is

$$J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(k+\nu+1)} \left(\frac{x}{2}\right)^{2k+\nu}$$

This function has a simpler form when ν is an odd half-integer. For example,

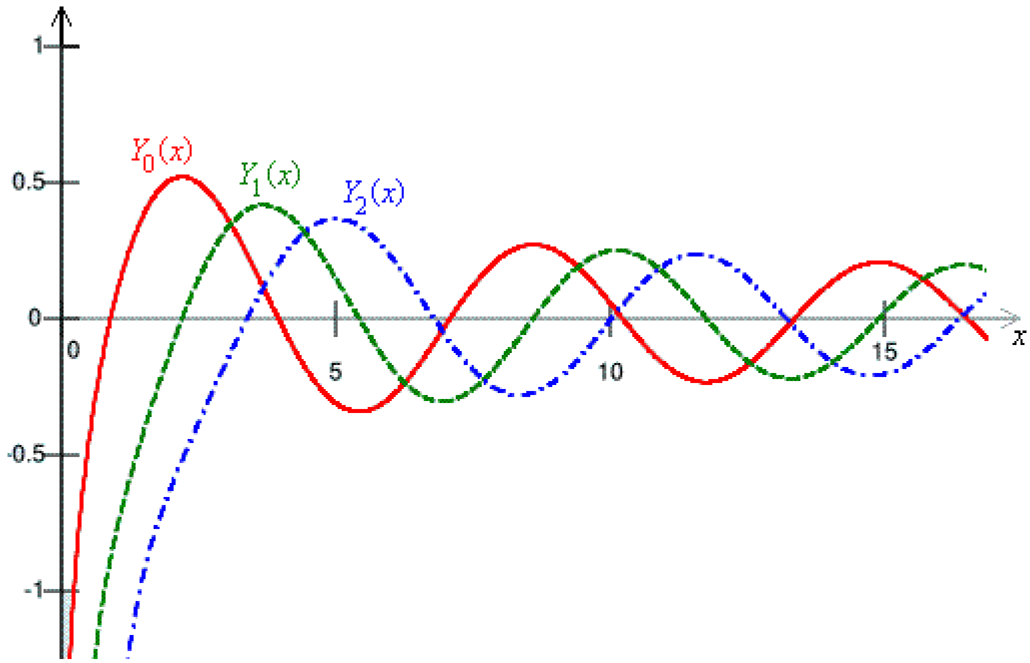
$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x \,, \qquad J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$$

The Bessel function of the second kind is

$$Y_\nu(x) = \frac{J_\nu(x)\cos(\nu\pi) - J_{-\nu}(x)}{\sin(\nu\pi)}$$

$Y_\nu(x)$ is unbounded as $x \to 0$: $\lim_{x\to 0^+} Y_\nu(x) = -\infty$

Bessel functions of the second kind (all of which have a singularity at $x = 0$):



Bessel functions arise frequently in situations where cylindrical or spherical polar coordinates are used.

A generalised Bessel ODE is

$$x^2\frac{d^2y}{dx^2} + (1-2a)x\frac{dy}{dx} + \left(b^2c^2x^{2c} + \left(a^2 - c^2\nu^2\right)\right)y = 0$$

whose general solution is

$$y(x) = x^a\left(A\,J_\nu\left(bx^c\right) + B\,Y_\nu\left(bx^c\right)\right)$$

For a generalised Bessel ODE with $a \ge 0$, whenever the solution must remain bounded as $x \to 0$, the general solution simplifies to $y(x) = A\,x^a J_\nu\left(bx^c\right)$.

Example 1.12.2

Find a Maclaurin series solution to Legendre's ODE

$$\left(1-x^2\right)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + p\left(p+1\right)y = 0$$

in the case when p is a non-negative integer.

$P(x)=1-x^2 \quad \Rightarrow \quad P(0)=1\neq 0 \quad \Rightarrow \quad x=0$ is a regular point of the ODE.

Let the general solution be $y(x) = \sum_{n=0}^{\infty} a_n x^n$.

Then $y'(x) = \sum_{n=1}^{\infty} n\,a_n x^{n-1}$ and $y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$.

Substitute into the ODE:

$$\sum_{n=0}^{\infty} n(n-1)a_n\left(x^{n-2}-x^n\right) - \sum_{n=0}^{\infty} 2n\,a_n x^n + \sum_{n=0}^{\infty} p(p+1)a_n x^n = 0$$

$$\Rightarrow \quad \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty}\left(-n(n-1) - 2n + p(p+1)\right)a_n x^n = 0$$

But the first two terms (n = 0 and n = 1) of the first series are both zero.

$$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} = 0 + 0 + \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

after a shift in indices. Returning to the full ODE,

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty}\left(-n^2+n-2n + p^2+p\right)a_n x^n = 0$$

$$\Rightarrow \quad \sum_{n=0}^{\infty}\left((n+2)(n+1)a_{n+2} + \left(-\left(n^2-p^2\right) - (n-p)\right)a_n\right)x^n = 0$$

Matching coefficients of x^n, $(n\geq 0)$:

$$(n+2)(n+1)a_{n+2} = \left(n^2-p^2+n-p\right)a_n \quad \Rightarrow \quad a_{n+2} = \frac{(n-p)(n+p+1)}{(n+2)(n+1)}a_n$$

Shifting indices, $a_n = \dfrac{(n+p-1)(n-p-2)}{n(n-1)}a_{n-2} \quad (n\geq 2)$, with a_0 and a_1 arbitrary.

Rearranging slightly, $a_n = \dfrac{-\left(p-(n-2)\right)\left(p+(n-1)\right)}{n(n-1)}a_{n-2} \quad (n\geq 2)$

Example 1.12.2 (continued)

$$\Rightarrow \quad a_2 = \frac{-p(p+1)}{2\times 1} a_0 \,, \quad a_3 = \frac{-(p-1)(p+2)}{3\times 2} a_1 \,,$$

$$a_4 = \frac{-(p-2)(p+3)}{4\times 3} a_2 = \frac{+p(p+1)(p-2)(p+3)}{4\times 3\times 2\times 1} a_0 \,,$$

$$a_5 = \frac{-(p-3)(p+4)}{5\times 4} a_3 = \frac{+(p-1)(p+2)(p-3)(p+4)}{5\times 4\times 3\times 2\times 1} a_1 \,,$$

$$a_6 = \frac{-(p-4)(p+5)}{6\times 5} a_4 = \frac{-p(p+1)(p-2)(p+3)(p-4)(p+5)}{6\times 5\times 4\times 3\times 2\times 1} a_0 \,, \quad \text{etc.}$$

It then follows that the general solution to Legendre's ODE is

$$y_p(x) = a_0 \left(1 - \frac{p(p+1)}{2!} x^2 + \frac{p(p+1)(p-2)(p+3)}{4!} x^4 - \frac{p(p+1)(p-2)(p+3)(p-4)(p+5)}{6!} x^6 + \dots \right)$$
$$+ a_1 \left(x - \frac{(p-1)(p+2)}{3!} x^3 + \frac{(p-1)(p+2)(p-3)(p+4)}{5!} x^5 - \frac{(p-1)(p+2)(p-3)(p+4)(p-5)(p+6)}{7!} x^7 + \dots \right)$$

where a_0 and a_1 are arbitrary constants. This series converges on [–1, 1] .

These solutions $y(x) = \sum_{n=0}^{\infty} a_n x^n$ are Legendre functions of order p .
If p is a non-negative integer then

$$a_{p+2} = \frac{(p-p)(p+p+1)}{(p+2)(p+1)} a_p = 0 \quad \Rightarrow \quad a_{p+4} = 0 \quad \Rightarrow \quad \dots$$

$$\Rightarrow \quad a_{p+2k} = 0 \quad \forall k \in \mathbb{N}$$

If we set $a_1 = 0$ when p is even, then the series solution terminates as a p^{th} order polynomial (and therefore converges for all x).

If we set $a_0 = 0$ when p is odd, then the series solution terminates as a p^{th} order polynomial (and therefore converges for all x).

With suitable choices of a_0 and a_1, so that $P_n(1) = 1$,
we have the set of **Legendre polynomials**:
$P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = \frac{1}{2}(3x^2 - 1)$,
$P_3(x) = \frac{1}{2}(5x^3 - 3x)$, $P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$, $P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$,
$P_6(x) = \frac{1}{16}(231x^6 - 315x^4 + 105x^2 - 5)$, etc.
Each $P_n(x)$ is a solution of Legendre's ODE with $p = n$.

Rodrigues' formula generates all of the Legendre polynomials:

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}\left(\left(x^2 - 1\right)^n \right)$$

Among the properties of Legendre polynomials is their orthogonality on [–1, 1]:

$$\int_{-1}^{+1} P_m(x) P_n(x)\, dx = \begin{cases} 0 & (m \neq n) \\ \dfrac{2}{2n+1} & (m = n) \end{cases}$$

[Space for any additional notes]

END OF CHAPTER 1