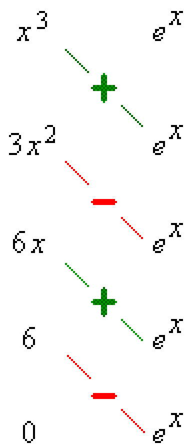
Examples of Integration by Parts

The method of integration by parts will be required in the next example of a first order linear ODE (Example 1.04.4). There are three main cases for integration by parts:

Example 1.04.2

Integrate $x^3 e^x$ with respect to *x*.

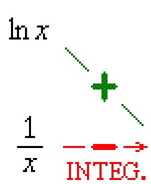


Therefore $$\int x^3 e^x \, dx = e^x \left(x^3 - 3x^2 + 6x - 6 \right) + C$$

This is an example where the table stops at a zero in the left column.

Example 1.04.3

Integrate $\ln x$ with respect to *x*.



Therefore $$\int \ln x \, dx = x \ln x - \int \frac{x}{x} dx = x \ln x - \int 1 \, dx = x \ln x - x + C$$

$$\Rightarrow \quad \int \ln x \, dx = x(\ln x - 1) + C$$

This is an example where the table stops at a row that can be integrated easily.

The third case, where the table stops at a row that is a multiple of the original integrand, follows in Example 1.04.4.

Example 1.04.4

An electrical circuit that contains a resistor, R = 8 Ω (ohm), an inductor, L = 0.02 millihenry, and an applied emf, $E(t) = 2\cos(5t)$, is governed by the differential equation

$$L\frac{di}{dt} + Ri = \frac{dE}{dt}$$

Determine the current at any time $t \geq 0$, if initially there is a current of 1 ampere in the circuit.

First note that the inductance $L = 2\times10^{-5}$ H is very small. The ODE is therefore not very different from

$$0 + R\,i = dE/dt$$

which has the immediate solution

$$i = (1/R)\,dE/dt = (1/8)\times(-10\sin 5t)$$

We therefore anticipate that $i = -(5/4)\sin 5t$ will be a good approximation to the exact solution.

Substituting all values ($R = 8$, $L = 2\times10^{-5}$, $E = 2\cos 5t \Rightarrow E' = -10\sin 5t$) into the ODE yields

$$\frac{di}{dt} + 4\times10^5\, i = -5\times10^5 \sin 5t$$

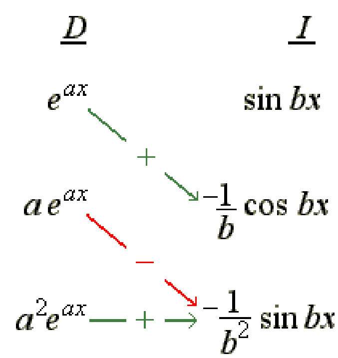
which is a linear first order ODE.

$P(t) = 400\,000$ and $R(t) = -500\,000\sin 5t \Rightarrow h = \int P\,dt = 400\,000\,t$

$\Rightarrow$ integrating factor $= e^h = e^{400\,000t}$

$\Rightarrow \int e^h R\,dt = -500\,000\int e^{400\,000t}\sin 5t\,dt$

Integration by parts of the general case $\int e^{ax}\sin bx\,dx$:



$$\Rightarrow \int e^{ax}\sin bx\,dx = \left[-\frac{1}{b}e^{ax}\cos bx + \frac{a}{b^2}e^{ax}\sin bx\right] - \int\frac{a^2}{b^2}e^{ax}\sin bx\,dx$$

$$= \frac{1}{b^2}\left[e^{ax}\left(-b\cos bx + a\sin bx\right)\right] - \frac{a^2}{b^2}\int e^{ax}\sin bx\,dx$$

Example 1.04.4 (continued)

$$\Rightarrow \quad \left(1+\frac{a^2}{b^2}\right)\int e^{ax}\sin bx\,dx \;=\; \frac{1}{b^2}\left[e^{ax}\left(a\sin bx \;-\; b\cos bx\right)\right]$$

$$\Rightarrow \quad \int e^{ax}\sin bx\,dx \;=\; \frac{1}{a^2+b^2}\left[e^{ax}\left(a\sin bx \;-\; b\cos bx\right)\right] \;+\; C$$

Set $a = 400\,000$, $b = 5$ and $x = t$:

$$\Rightarrow \quad \int e^{h}R\,dt \;=\; -500\,000\,\frac{1}{400\,000^2+5^2}\,e^{400000t}\left(400\,000\sin 5t - 5\cos 5t\right)$$

The general solution is

$$i(t) \;=\; e^{-h}\left(\int e^{h}R\,dt \;+\; C\right)$$

$$\Rightarrow \quad i(t) \;=\; A\,e^{-400000t} \;-\; \frac{500\,000}{400\,000^2+25}\left(400\,000\sin 5t - 5\cos 5t\right)$$

But $i(0) = 1$

$$\Rightarrow \quad 1 \;=\; A \;-\; \frac{500\,000}{400\,000^2+25}\left(0 \;-\; 5\right)$$

$$\Rightarrow \quad A \;=\; (400\,000^2 + 25 - 2\,500\,000) \,/\, (400\,000^2 + 25)$$

Therefore the complete solution is [exactly]

$$\boxed{i(t) \;=\; \frac{159\,997\,500\,025\,e^{-400000\,t} \;-\; 500\,000\left(400\,000\sin 5t \;-\; 5\cos 5t\right)}{160\,000\,000\,025}}$$

To an excellent approximation, this complete solution is

$$\Rightarrow \quad i(t) \;\approx\; e^{-400000t} \;-\; \frac{5}{4}\sin 5t$$

After only a few microseconds, the transient term is negligible.
The complete solution is then, to an excellent approximation,

$$i(t) \;\approx\; -\;\frac{5}{4}\sin 5t$$

as before.

1.05 Bernoulli ODEs

The first order linear ODE is a special case of the Bernoulli ODE

$$\frac{dy}{dx} + P(x)\,y = R(x)\,y^n$$

If $n = 0$ then the ODE is linear.
If $n = 1$ then the ODE is separable.

For any other value of n, the change of variables $u = \frac{y^{1-n}}{1-n}$ will convert the Bernoulli ODE for y into a linear ODE for u.

$$\frac{du}{dx} = \frac{du}{dy}\frac{dy}{dx} = \frac{1-n}{1-n}y^{-n}\frac{dy}{dx} \quad \Rightarrow \quad \frac{dy}{dx} = y^n\frac{du}{dx}$$

The ODE transforms to

$$y^n\frac{du}{dx} + P(x)\,y = R(x)\,y^n \quad \Rightarrow \quad \frac{du}{dx} + P(x)\,y^{1-n} = R(x)$$

We therefore obtain the linear ODE for u:

$$\frac{du}{dx} + \big((1-n)P(x)\big)u = R(x)$$

whose solution is

$$\boxed{\frac{y^{1-n}}{1-n} = u(x) = e^{-h(x)}\left(\int e^{h(x)}R(x)\,dx + C\right), \quad \text{where} \quad h(x) = (1-n)\int P(x)\,dx}$$

together with the singular solution $y \equiv 0$ in the cases where $n > 0$.

Example 1.05.1

Find the general solution of the logistic population model

$$\frac{dy}{dx} = ay - by^2$$

where a, b are positive constants.

The Bernoulli equation is

$$\frac{dy}{dx} + (-a)y = (-b)y^2$$

with $P = -a, \quad R = -b, \quad n = 2.$

$$h = (1-n)\int P\,dx = (-1)\int -a\,dx = ax$$

Integrating factor $e^h = e^{ax}$

$$\int e^h R\,dx = \int e^{ax}(-b)\,dx = -\frac{b}{a}e^{ax} \qquad \text{(Note that } a > 0\text{)}$$

$$\frac{y^{-1}}{-1} = u = e^{-h}\left(\int e^h R\,dx + C\right) = e^{-ax}\left(-\frac{b}{a}e^{ax} + C\right)$$

$$\Rightarrow \quad y = \frac{a}{b - Ae^{-ax}}$$

Note that

$$y(0) = \frac{a}{b-A} \quad \Rightarrow \quad A = b - \frac{a}{y(0)} \qquad \text{and} \qquad \lim_{x\to\infty} y = \frac{a}{b}$$

Also $y \equiv 0$ is a solution to the original ODE that is not included in the above solution for any finite value of the arbitrary constant A.

The general solution is

$$\boxed{y = \frac{a}{b - Ae^{-ax}} \quad \text{or} \quad y \equiv 0}$$

[Note that the initial condition is not positive and there is a discontinuity in y at $x = \frac{1}{a}\ln\frac{A}{b}$ if $A \geq b$ is true.]