1.09 Laplace Transforms

Laplace transforms can convert some initial value problems into algebra problems. It is assumed here that students have met Laplace transforms before. Only the key results are displayed here, before they are employed to solve some initial value problems.

The Laplace transform of a function $f(t)$ is the integral

$$\boxed{F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt}$$

where the integral exists.

Some standard transforms and properties are:

Linearity:

$$\mathcal{L}\{a\,f(t) + b\,g(t)\} = a\,\mathcal{L}\{f(t)\} + b\,\mathcal{L}\{g(t)\} \qquad (a, b = \text{constants})$$

Polynomial functions:

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}} \quad \Rightarrow \quad \mathcal{L}^{-1}\left\{\frac{1}{s^n}\right\} = \frac{t^{n-1}}{(n-1)!}$$

First Shift Theorem:

$$\mathcal{L}\{f(t)\} = F(s) \quad \Rightarrow \quad \mathcal{L}\{e^{at} f(t)\} = F(s-a)$$

$$\Rightarrow \quad \mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at} \quad \text{and} \quad \mathcal{L}^{-1}\left\{\frac{1}{(s-a)^n}\right\} = \frac{t^{n-1}e^{at}}{(n-1)!}$$

Trigonometric Functions:

$$\mathcal{L}\{e^{at}\sin\omega t\} = \frac{\omega}{(s-a)^2+\omega^2} \quad \Rightarrow \quad \mathcal{L}^{-1}\left\{\frac{1}{(s-a)^2+\omega^2}\right\} = \frac{e^{at}\sin\omega t}{\omega}$$

$$\mathcal{L}\{e^{at}\cos\omega t\} = \frac{s-a}{(s-a)^2+\omega^2} \quad \Rightarrow \quad \mathcal{L}^{-1}\left\{\frac{s-a}{(s-a)^2+\omega^2}\right\} = e^{at}\cos\omega t$$

Derivatives:

$$\mathcal{L}\{f'(t)\} = s\,\mathcal{L}\{f(t)\} - f(0)$$

$$\mathcal{L}\{f''(t)\} = s^2\,\mathcal{L}\{f(t)\} - s\,f(0) - f'(0)$$

Integration:

$$\mathscr{L}^{-1}\left\{\frac{1}{s}G(s)\right\} = \int_0^t \mathscr{L}^{-1}\{G(s)\}\,d\tau$$

$$\frac{d}{ds}\mathscr{L}\{f(t)\} = -\mathscr{L}\{t\,f(t)\} \quad \Rightarrow \quad \mathscr{L}^{-1}\{F'(s)\} = -t\cdot\mathscr{L}^{-1}\{F(s)\}$$

Second shift theorem:

$$\mathscr{L}^{-1}\{F(s)\} = f(t) \qquad \Rightarrow \qquad \mathscr{L}^{-1}\{e^{-as}F(s)\} = H(t-a)\,f(t-a)$$

where $H(t-a) = \begin{cases} 0 & (t<a) \\ 1 & (t \ge a) \end{cases}$ is the Heaviside (unit step) function.

Dirac delta function

$$\mathscr{L}\{\delta(t-a)\} = e^{-as}$$

where $\displaystyle\int_c^d f(t)\delta(t-a)\,dt = \begin{cases} f(a) & (\text{if} \quad c<a<d) \\ 0 & (a<c \ \text{ or } \ a>d) \end{cases}$.

For a periodic function $f(t)$ with fundamental period p,

$$\mathscr{L}\{f(t)\} = \frac{1}{1-e^{-ps}}\int_0^p e^{-st} f(t)\,dt$$

Convolution:

$$\mathscr{L}^{-1}\{F(s)G(s)\} = \mathscr{L}^{-1}\{F(s)\} * \mathscr{L}^{-1}\{G(s)\}$$

where $(f*g)(t)$ denotes the convolution of $f(t)$ and $g(t)$ and is defined by

$$(f*g)(t) = \int_0^t f(\tau)\,g(t-\tau)\,d\tau$$

The identity function for convolution is the Dirac delta function:

$$\delta(t-a)*f(t) = f(t-a)H(t-a) \quad \Rightarrow \quad \delta(t)*f(t) = f(t)$$

Here is a summary of inverse Laplace transforms.

$F(s)$	$f(t)$
$\int_0^\infty e^{-st} f(t)\, dt$	$f(t)$
$\dfrac{1}{s^n} \quad (n \in \mathbb{N})$	$\dfrac{t^{n-1}}{(n-1)!}$
$\dfrac{1}{\sqrt{s}}$	$\dfrac{1}{\sqrt{\pi t}}$
$\dfrac{1}{s-a}$	e^{at}
$\dfrac{1}{(s-a)^n} \quad (n \in \mathbb{N})$	$\dfrac{t^{n-1}\, e^{at}}{(n-1)!}$
e^{-as}	$\delta(t-a)$
$\dfrac{e^{-as}}{s}$	$H(t-a)$
$\dfrac{1}{s^2+\omega^2}$	$\dfrac{\sin \omega t}{\omega}$
$\dfrac{1}{(s-a)^2+\omega^2}$	$\dfrac{e^{at} \sin \omega t}{\omega}$
$\dfrac{1}{(s-a)^2-b^2}$	$\dfrac{e^{at} \sinh bt}{b}$
$\dfrac{(s-a)}{(s-a)^2+\omega^2}$	$e^{at} \cos \omega t$
$\dfrac{(s-a)}{(s-a)^2-b^2}$	$e^{at} \cosh bt$

$F(s)$	$f(t)$
$\dfrac{1}{s(s^2+\omega^2)}$	$\dfrac{1-\cos \omega t}{\omega^2}$
$\dfrac{1}{s^2(s^2+\omega^2)}$	$\dfrac{\omega t - \sin \omega t}{\omega^3}$
$\dfrac{1}{(s^2+\omega^2)^2}$	$\dfrac{\sin \omega t - \omega t \cos \omega t}{2\,\omega^3}$
$\dfrac{s}{(s^2+\omega^2)^2}$	$\dfrac{t \sin \omega t}{2\,\omega}$
$\dfrac{s^2-\omega^2}{(s^2+\omega^2)^2}$	$t \cos \omega t$
$\dfrac{1}{s} \tanh\left(\dfrac{as}{2}\right)$	Square wave, period $2a$, amplitude 1
$\dfrac{1}{s^2} \tanh\left(\dfrac{as}{2}\right)$	Triangular wave, period $2a$, amplitude a
$\dfrac{b}{as^2} - \dfrac{b}{s(e^{as}-1)}$	Sawtooth wave, period a, amplitude b
$\{ s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - s^{n-3} f''(0) - \ldots - s f^{(n-2)}(0) - f^{(n-1)}(0) \}$	$\dfrac{d^n f}{dt^n}$
$\dfrac{1}{s} F(s)$	$\int_0^t f(\tau)\, d\tau$
$\dfrac{dF}{ds}$	$-t\, f(t)$

Example 1.09.1 (Example 1.08.2 again)

Find the general solution of the ODE

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = e^{-2x}$$

The initial conditions are unknown, so let $a = y(0)$ and $b = y'(0)$.
Taking the Laplace transform of the initial value problem,

$$\left(s^2Y - s\cdot a - b\right) + 4\left(sY - a\right) + 4Y = \frac{1}{s+2}$$

$$\Rightarrow \quad \left(s^2+4s+4\right)Y = as+4a+b + \frac{1}{s+2}$$

$$\Rightarrow \quad Y = \frac{as+4a+b}{(s+2)^2} + \frac{1}{(s+2)^3} = a\frac{(s+2)}{(s+2)^2} + \frac{2a+b}{(s+2)^2} + \frac{1}{(s+2)^3}$$

Note that $\mathscr{L}^{-1}\left\{\frac{1}{(s+a)^n}\right\} = \frac{x^{n-1}e^{-ax}}{(n-1)!}$

$$\Rightarrow \quad y = a\,\mathscr{L}^{-1}\left\{\frac{1}{s+2}\right\} + (2a+b)\mathscr{L}^{-1}\left\{\frac{1}{(s+2)^2}\right\} + \mathscr{L}^{-1}\left\{\frac{1}{(s+2)^3}\right\}$$

$$\Rightarrow \quad y = a\,e^{-2x} + (2a+b)\,x\,e^{-2x} + \frac{x^2e^{-2x}}{2!}$$

Introducing the new arbitrary constants $A = 2a + b$ and $B = a$, we recover the general solution

$$\boxed{y = \left(\tfrac{1}{2}x^2 + Ax + B\right)e^{-2x}}$$

Example 1.09.2 (Example 1.07.1 again)

Find the complete solution to the initial value problem

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 26x = 10\,\delta(t-3)$$

(where $\delta(t-a)$ is the Dirac delta function),
together with the initial conditions $x(0) = x'(0) = 0$.

Let $X(s) = \mathscr{L}\{x(t)\}$ be the Laplace transform of the solution $x(t)$.
Taking the Laplace transform of the initial value problem,

$$\left(s^2 X - 0 - 0\right) + 2\left(s X - 0\right) + 26X = 10e^{-3s}$$

$$\Rightarrow \quad \left(s^2+2s+26\right)X = 10e^{-3s} \qquad \Rightarrow \quad X = \frac{10e^{-3s}}{(s+1)^2+5^2} = \mathscr{L}\left\{2e^{-t}\sin 5t\right\}e^{-3s}$$

By the second shift theorem, it then follows that the complete solution is

$$x(t) = 2e^{-t}\sin 5t\Big|_{t\to t-3} H(t-3) = 2e^{-(t-3)}\sin 5(t-3)\,H(t-3)$$

This is a considerably faster solution than that provided by the method of variation of parameters (Example 1.07.1).

OR

$$X(s) = \frac{10e^{-3s}}{(s+1)^2+5^2} = \mathscr{L}\left\{2e^{-t}\sin 5t\right\}\cdot\mathscr{L}\left\{\delta(t-3)\right\} = \mathscr{L}\left\{2e^{-t}\sin 5t * \delta(t-3)\right\}$$

Using the convolution properties of the Dirac delta function,

$$x(t) = \left(2e^{-t}\sin 5t\right) * \delta(t-3) = 2e^{-(t-3)}\sin 5(t-3)\,H(t-3)$$

1.10 Series Solutions of ODEs

If the functions $p(x)$, $q(x)$ and $r(x)$ in the ODE

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = r(x)$$

are all analytic in some interval $x_o - h < x < x_o + h$ (and therefore possess Taylor series expansions around x_o with radii of convergence of at least h), then a series solution to the ODE around x_o with a radius of convergence of at least h exists:

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_o)^n, \qquad a_n = \frac{y^{(n)}(x_o)}{n!}$$

Example 1.10.1

Find a series solution as far as the term in x^3, to the initial value problem

$$\frac{d^2y}{dx^2} - x\frac{dy}{dx} + e^x y = 4; \qquad y(0) = 1, \quad y'(0) = 4$$

None of our previous methods apply to this problem.
The functions $-x$, e^x and 4 are all analytic everywhere.

The solution of this ODE, expressed as a power series, is

$$y(x) = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \ldots$$

But $y(0) = 1$ and $y'(0) = 4$.
From the ODE,

$$y'' = x\,y' - e^x y + 4 \quad \Rightarrow \quad y''(0) = 0 - y(0) + 4 = -1 + 4 = 3$$

Differentiating the ODE,

$$y''' = y' + x\,y'' - e^x y - e^x y' + 0 \quad \Rightarrow \quad y'''(0) = y'(0) + 0 - y(0) - y'(0) = -1$$

Therefore the first four terms of the solution are

$$\boxed{y(x) = 1 + 4x + \frac{3}{2}x^2 - \frac{1}{6}x^3 + \ldots}$$

Example 1.10.2

Find the general solution (as a power series about $x = 0$) to the ordinary differential equation

$$\frac{d^2 y}{dx^2} + x^2 y = 0$$

Let the general solution be $y(x) = \sum_{n=0}^{\infty} a_n x^n$.

Then $y'(x) = \sum_{n=1}^{\infty} n\, a_n x^{n-1}$ and $y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$.

Substitute into the ODE:

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + x^2 \sum_{n=0}^{\infty} a_n x^n = 0 \quad \Rightarrow \quad \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^{n+2} = 0$$

Shift the indices on each summation so that the exponent of x is n in both cases:

$$\Rightarrow \quad \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=2}^{\infty} a_{n-2} x^n = 0$$

Bring the two summations together for all terms from $n = 2$ onwards:

$$\Rightarrow \quad 2\times 1 a_2 x^0 + 3\times 2 a_3 x^1 + \sum_{n=2}^{\infty} \left((n+2)(n+1) a_{n+2} + a_{n-2} \right) x^n = 0$$

But this equation must be true regardless of the choice of x.

Therefore the coefficient of each power of x must be zero. $\Rightarrow \quad a_2 = a_3 = 0$

and $\quad a_{n+2} = \dfrac{-a_{n-2}}{(n+2)(n+1)} \quad (n = 2, 3, 4, \ldots) \quad \Rightarrow \quad a_n = \dfrac{-a_{n-4}}{n(n-1)} \quad (n = 4, 5, 6, \ldots)$

$$\Rightarrow \quad a_4 = \frac{-a_0}{4\times 3} = -\frac{a_0}{12}, \qquad a_5 = \frac{-a_1}{5\times 4} = -\frac{a_1}{20}$$

$$a_6 = \frac{-a_2}{6\times 5} = 0, \qquad a_7 = \frac{-a_3}{7\times 6} = 0$$

$$a_8 = \frac{-a_4}{8\times 7} = \frac{+a_0}{56\times 12} = \frac{+a_0}{672}, \qquad a_9 = \frac{-a_5}{9\times 8} = \frac{+a_1}{72\times 20} = \frac{+a_1}{1440}, \quad \ldots$$

and $a_0 = y(0)$ and $a_1 = y'(0)$ are arbitrary.

Therefore the general solution is

$$y(x) = a_0 + a_1 x + 0x^2 + 0x^3 - \frac{a_0}{12} x^4 - \frac{a_1}{20} x^5 + 0x^6 + 0x^7$$

$$+ \frac{a_0}{672} x^8 + \frac{a_1}{1440} x^9 + 0x^{10} + 0x^{11} + \ldots$$

or

$$\boxed{y(x) = a_0\left(1 - \tfrac{1}{12} x^4 + \tfrac{1}{672} x^8 - \ldots\right) + a_1\left(x - \tfrac{1}{20} x^5 + \tfrac{1}{1440} x^9 - \ldots\right)}$$

1.11 The Gamma Function

The gamma function $\Gamma(x)$ is a special function that will be needed in the solution of Bessel's ODE. $\Gamma(x)$ is a generalisation of the factorial function $n!$ from positive integers to most real numbers. For any positive integer n,
$n! = n \times (n-1) \times (n-2) \times \ldots \times 3 \times 2 \times 1$ (with $0!$ defined to be 1)

When x is a positive integer n, $\boxed{\Gamma(n) = (n-1)!}$

We know that $n! = n \times (n-1)!$
The gamma function has a similar recurrence relationship: $\Gamma(x+1) = x \cdot \Gamma(x)$

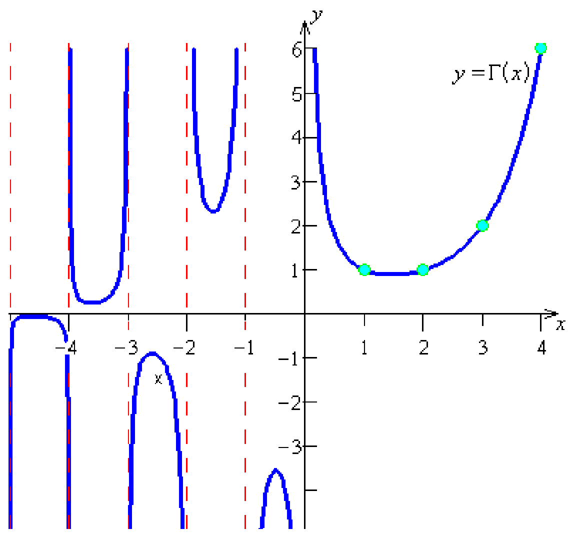
This allows $\Gamma(x)$ to be defined for non-integer negative x, using $\boxed{\Gamma(x) = \dfrac{\Gamma(x+1)}{x}}$

For example,
it can be shown that $\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$

$$\Rightarrow \quad \Gamma\left(-\tfrac{1}{2}\right) = \frac{\Gamma\left(+\frac{1}{2}\right)}{-\frac{1}{2}} = -2\sqrt{\pi} \qquad \Rightarrow \quad \Gamma\left(-\tfrac{3}{2}\right) = \frac{\Gamma\left(-\frac{1}{2}\right)}{-\frac{3}{2}} = +\frac{4\sqrt{\pi}}{3}, \text{ etc.}$$

$\Gamma(x)$ is infinite when x is a negative integer or zero. It is well defined for all other real numbers x.

In this graph of $y = \Gamma(x)$, values of the factorial function (at positive integer values of x) are highlighted.



There are several ways to define the gamma function, such as

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt \qquad (x > 0)$$

and

$$\Gamma(x) = \lim_{n \to \infty} \frac{n!\, n^x}{x(x+1)\ldots(x+n)}$$

A related special function is the **beta function**:

$$B(m,n) = \int_0^1 t^{m-1}(1-t)^{n-1}\,dt = 2\int_0^{\pi/2} \sin^{2m-1}\theta\,\cos^{2n-1}\theta\,d\theta = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

Among the many results involving the gamma function are:

For the closed region V in the first octant, bounded by the coordinate planes and the surface $\left(\frac{x}{a}\right)^{\alpha} + \left(\frac{y}{b}\right)^{\beta} + \left(\frac{z}{c}\right)^{\gamma} = 1$, with all constants positive,

$$I = \iiint_V x^{p-1}y^{q-1}z^{r-1}\,dx\,dy\,dz = \frac{a^p b^q c^r}{\alpha\beta\gamma}\cdot\frac{\Gamma\left(\frac{p}{\alpha}\right)\Gamma\left(\frac{q}{\beta}\right)\Gamma\left(\frac{r}{\gamma}\right)}{\Gamma\left(1+\frac{p}{\alpha}+\frac{q}{\beta}+\frac{r}{\gamma}\right)}$$

For the closed area A in the first quadrant, bounded by the coordinate axes and the curve $\left(\frac{x}{a}\right)^{\alpha} + \left(\frac{y}{b}\right)^{\beta} = 1$, with all constants positive,

$$I = \iint_A x^{p-1}y^{q-1}\,dx\,dy = \frac{a^p b^q}{\alpha\beta}\cdot\frac{\Gamma\left(\frac{p}{\alpha}\right)\Gamma\left(\frac{q}{\beta}\right)}{\Gamma\left(1+\frac{p}{\alpha}+\frac{q}{\beta}\right)}$$

Example 1.11.1

Establish the formula for the area enclosed by an ellipse.

The Cartesian equation of a standard ellipse is $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$.

Set $\alpha = \beta = 2$ and $p = q = 1$, then

$$A = 4I = 4\iint_A 1\,dx\,dy = 4\frac{a^1b^1}{2\times 2}\cdot\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(1+\frac{1}{2}+\frac{1}{2}\right)} = ab\cdot\frac{\left(\Gamma\left(\frac{1}{2}\right)\right)^2}{\Gamma(2)} = ab\frac{\left(\sqrt{\pi}\right)^2}{1!} = \pi ab$$