2. Matrix Algebra

A linear system of m equations in n unknowns,

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned}$$

(where the a_{ij} and b_i are constants)

can be written more concisely in matrix form, as

$$A\vec{\mathbf{x}} = \vec{\mathbf{b}}$$

where the ($m \times n$) coefficient matrix [m rows and n columns] is

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}$$

and the column vectors (also ($n \times 1$) and ($m \times 1$) matrices respectively) are

$$\vec{\mathbf{x}} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \text{and} \quad \vec{\mathbf{b}} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Matrix operations can render the solution of a linear system much more efficient.

Sections in this Chapter

2.01 Gaussian Elimination

Example 2.01.1

In quantum mechanics, the Planck length L_P is defined in terms of three fundamental constants:

- the universal constant of gravitation, $G = 6.67 \times 10^{-11}$ N m^2 kg^{-2}
- Planck's constant, $h = 6.62 \times 10^{-34}$ J s
- the speed of light in a vacuum, $c = 2.998 \times 10^{8}$ m s^{-1}

The Planck length is therefore

$$L_P = k\,G^x h^y c^z$$

where k is a dimensionless constant and x, y, z are constants to be determined.
Also note that 1 N = 1 kg m s^{-2} and 1 J = 1 Nm = 1 kg m^2 s^{-2} .
Use dimensional analysis to find the values of x, y and z .

Let $\left[L_P\right]$ denote the dimensions of L_P.

Then $\left[L_P\right] = \left[k\,G^x h^y c^z\right] = \left[G\right]^x \left[h\right]^y \left[c\right]^z = \left(\text{kg}^{-1}\text{m}^3\text{s}^{-2}\right)^x \left(\text{kg m}^2\text{s}^{-1}\right)^y \left(\text{m s}^{-1}\right)^z$

$$= \text{kg}^{-x+y}\text{m}^{3x+2y+z}\text{s}^{-2x-y-z} = \left[L_P\right] = \text{m}^1$$

This generates a linear system of three simultaneous equations for the three unknowns,

kg: $-x + y = 0$
m: $3x + 2y + z = 1$
s: $-2x - y - z = 0$

This can be re-written as the matrix equation **Ax = b** , where

$$\text{A} = \begin{bmatrix} -1 & 1 & 0 \\ 3 & 2 & 1 \\ -2 & -1 & -1 \end{bmatrix}, \quad \vec{\mathbf{x}} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \vec{\mathbf{b}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Use Gaussian elimination (a sequence of row operations) on the augmented matrix [A | **b**] :

$$\left[\text{A}\,|\,\vec{\mathbf{b}}\right] = \left[\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 3 & 2 & 1 & 1 \\ -2 & -1 & -1 & 0 \end{array}\right]$$

Multiply Row 1 by (–1):

$$\xrightarrow{R_1 \times -1} \left[\begin{array}{ccc|c} \boxed{1} & -1 & 0 & 0 \\ 3 & 2 & 1 & 1 \\ -2 & -1 & -1 & 0 \end{array}\right]$$

There is now a "leading one" in the top left corner.

Example 2.01.1 (continued)

From Row 2 subtract (3 × Row 1) and
to Row 3 add (2 × Row 1):

$$\xrightarrow[R_3+2R_1]{R_2-3R_1} \left[\begin{array}{ccc|c} \boxed{1} & -1 & 0 & 0 \\ 0 & 5 & 1 & 1 \\ 0 & -3 & -1 & 0 \end{array}\right]$$

All entries below the first leading one are now zero.
The next leading entry is a '5'. Scale it down to a '1'.
Multiply Row 2 by (1/5):

$$\xrightarrow[R_2 \times \frac{1}{5}]{} \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & \boxed{1} & \frac{1}{5} & \frac{1}{5} \\ 0 & -3 & -1 & 0 \end{array}\right]$$

Clear the entry below the new leading one.
To Row 3 add (3 × Row 2):

$$\xrightarrow[R_3+3R_2]{} \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & \boxed{1} & \frac{1}{5} & \frac{1}{5} \\ 0 & 0 & -\frac{2}{5} & \frac{3}{5} \end{array}\right]$$

The next leading entry is a '–2/5'. Scale it down to a '1'.
Multiply Row 3 by (–5/2):

$$\xrightarrow[R_3 \times -\frac{5}{2}]{} \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & \frac{1}{5} & \frac{1}{5} \\ 0 & 0 & \boxed{1} & -\frac{3}{2} \end{array}\right]$$

This matrix is in **row echelon form** (the first non-zero entry in every row is a one and all entries below every leading one in its column are zero). It is also **upper triangular** (all entries below the leading diagonal are zero).
The solution may be read from the echelon form, using back substitution:

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & \frac{1}{5} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{5} \\ -\frac{3}{2} \end{bmatrix} \quad \Rightarrow \quad z = -\frac{3}{2} \quad \Rightarrow \quad y + \frac{1}{5} \times \left(-\frac{3}{2}\right) = \frac{1}{5} \quad \Rightarrow \quad y = \frac{1}{5} \times \frac{5}{2} = \frac{1}{2}$$

$$\Rightarrow \quad x - \frac{1}{2} = 0 \quad \Rightarrow \quad x = \frac{1}{2}$$

Example 2.01.1 (continued)

An alternative strategy is to complete the reduction of the augmented matrix to **reduced row echelon form** (the first non-zero entry in every row is a one and all other entries are zero in a column that contains a leading one).

From Row 2 subtract (1/5 × Row 3):

$$\xrightarrow[R_2 - \frac{1}{5}R_3]{} \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{3}{2} \end{array}\right]$$

To Row 1 add Row 2:

$$\xrightarrow{R_1 + R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{3}{2} \end{array}\right]$$

From this reduced row echelon matrix, the values of x, y and z may be read directly:

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{3}{2} \end{bmatrix} \quad \Rightarrow \quad x = y = \tfrac{1}{2}, \quad z = -\tfrac{3}{2}$$

When a square linear system (same number of equations as unknowns) has a unique solution, the reduced row echelon form of the coefficient matrix is the identity matrix.

Therefore the functional form of the Planck length is

$$L_P = k\sqrt{\frac{Gh}{c^3}} = \frac{k}{c}\sqrt{\frac{Gh}{c}}$$

Dimensional analysis alone cannot determine the value of the constant k.
[Methods in quantum mechanics, beyond the scope of this course, can establish that the constant is $k = \dfrac{1}{2\pi}$, so that $L_P = 1.616\,20 \times 10^{-35}$ m.]

Example 2.01.2

Find the solution (x, y, z, t) to the system of equations

$$\begin{aligned} x + y \qquad &= 5 \\ y + z \quad &= 7 \\ 2y + z + t &= 10 \end{aligned}$$

This is an **under-determined system** of equations (fewer equations than unknowns).
A unique solution is not possible. There will be either infinitely many solutions or no solution at all.

Reduce the augmented matrix to reduced row echelon form:

$$\left[\begin{array}{cccc|c} \boxed{1} & 1 & 0 & 0 & 5 \\ 0 & \boxed{1} & 1 & 0 & 7 \\ 0 & 2 & 1 & 1 & 10 \end{array}\right]$$

A leading one exists in the top left entry, with zero elsewhere in the first column.
A leading one exists in the second row. Clear the other entries in the second column.
From Row 3 subtract (2 × Row 2) and
from Row 1 subtract Row 2:

$$\xrightarrow[R_3 - 2R_2]{R_1 - R_2} \left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & -2 \\ 0 & \boxed{1} & 1 & 0 & 7 \\ 0 & 0 & -1 & 1 & -4 \end{array}\right]$$

Rescale the leading entry in Row 3 to a '1'.
Multiply row 3 by (–1):

$$\xrightarrow[R_3 \times -1]{} \left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & -2 \\ 0 & 1 & 1 & 0 & 7 \\ 0 & 0 & \boxed{1} & -1 & 4 \end{array}\right]$$

Clear the other entries in the third column.
From Row 2 subtract Row 3 and
to Row 1 add Row 3:

$$\xrightarrow[R_2 - R_3]{R_1 + R_3} \left[\begin{array}{cccc|c} \boxed{1} & 0 & 0 & -1 & 2 \\ 0 & \boxed{1} & 0 & 1 & 3 \\ 0 & 0 & \boxed{1} & -1 & 4 \end{array}\right]$$

The leading ones are identified in this row reduced echelon form.

Example 2.01.2 (continued)

$$\begin{bmatrix} \boxed{1} & 0 & 0 & -1 \\ 0 & \boxed{1} & 0 & 1 \\ 0 & 0 & \boxed{1} & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

The fourth column lacks a leading one. This means that the fourth variable, t, is a free parameter, in terms of which the other three variables may be expressed. We therefore have a **one-parameter family of solutions**,

$$x - t = 2 , \quad y + t = 3 , \quad z - t = 4$$

$\Rightarrow$ $x = 2 + t , \quad y = 3 - t , \quad z = 4 + t$

or

$$(x, y, z, t) = (2, 3, 4, 0) + (1, -1, 1, 1)\, t$$

where t is free to be any real number.

The **rank** of a matrix is the number of leading ones in its echelon form.

If rank (A) < rank [A | **b**] , then the linear system is **inconsistent** and has no solution.

If rank (A) = rank [A | **b**] = n (the number of columns in A), then the system has a unique solution for any such vector **b** .

If rank (A) = rank [A | **b**] < n , then the system has infinitely many solutions, with a number of parameters = (n – rank (A)) = (# columns in A_r with no leading one).

Example 2.01.3

Read the solution set $(x_1, x_2, \ldots, x_n)$ from the following reduced echelon forms.

(a)

$$\left[\begin{array}{cccc|c} \boxed{1} & 0 & -2 & 1 & 1 \\ 0 & \boxed{1} & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

rank (A) = rank [A | **b**] = 2 , $n = 4$

Two-parameter family of solutions:

$$(x_1, x_2, x_3, x_4) = (1, 2, 0, 0) + (2, -1, 1, 0)\, x_3 + (-1, 0, 0, 1)\, x_4$$

Example 2.01.3

(b)

$$\left[\begin{array}{cccc|c} \boxed{1} & 0 & -2 & 1 & 1 \\ 0 & \boxed{1} & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & \boxed{1} \end{array}\right] \qquad \text{rank (A)} < \text{rank } [\text{A} \mid \mathbf{b}] \quad \Rightarrow \quad \text{no solution}$$

(c)

$$\left[\begin{array}{ccc|c} \boxed{1} & 0 & 0 & a \\ 0 & \boxed{1} & 0 & b \\ 0 & 0 & \boxed{1} & c \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \text{rank (A)} = \text{rank } [\text{A} \mid \mathbf{b}] = n = 3 \quad \Rightarrow \quad \text{unique solution}$$

This is an **over-determined system** of five equations in three unknowns, but two of the five equations are superfluous and can be expressed in terms of the other three equations. In this case a unique solution exists regardless of the values of the numbers *a*, *b*, *c*.

The solution is

$$(x_1, x_2, x_3) = (a, b, c)$$

Note that software exists to eliminate the tedious arithmetic of the row operations. Various procedures exist in Maple and Matlab.

A custom program, available on the course web site at "`www.engr.mun.ca/~ggeorge/9420/demos/`", allows the user to enter the coefficients of a linear system as rational numbers, allows the user to perform row operations (but will *not* suggest the appropriate operation to use) and carries out the arithmetic of the chosen row operation automatically.