4. Stability Analysis for Non-linear Ordinary Differential Equations

A pair of simultaneous first order homogeneous linear ordinary differential equations for two functions $x(t)$, $y(t)$ of one independent variable t,

$$\dot{x} = \frac{dx}{dt} = ax + by$$

$$\dot{y} = \frac{dy}{dt} = cx + dy$$

may be represented by the matrix equation $\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

A single second order linear homogeneous ordinary differential equation for $x(t)$ with constant coefficients,

$$\frac{d^2x}{dt^2} + p\frac{dx}{dt} + qx = 0 \qquad \Rightarrow \quad \frac{dy}{dt} + py + qx = 0$$

may be re-written as a linked pair of first order homogeneous ordinary differential equations, by introducing a second dependent variable:

$$\frac{dx}{dt} = y \qquad \Rightarrow \quad \frac{dy}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2}$$

$$\frac{dy}{dt} = -qx - py$$

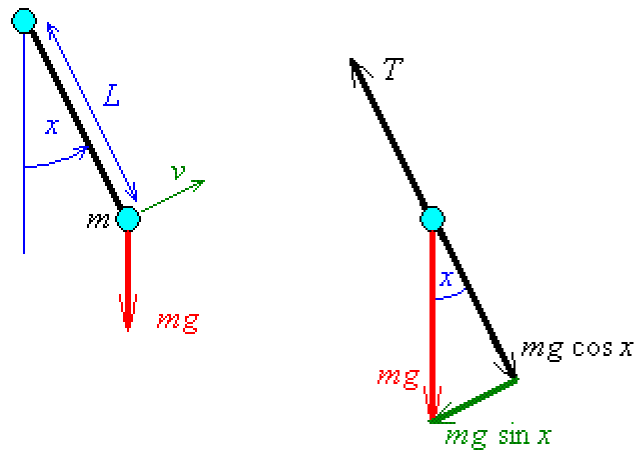
and may also be represented in matrix form $\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -q & -p \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

The general solution for (x, y) in either case can be displayed graphically as a set of contour curves (or level curves) in a **phase space**.

Sections in this chapter:

4.01 Motion of a Pendulum

Consider a pendulum, moving under its own weight, without friction.
The pendulum bob has mass *m*, the shaft has length *L* and negligible mass, and the angle of the shaft with the vertical is *x*. The tension along the shaft is *T*. The acceleration due to gravity is *g* (≈ 9.81 m s^{-2}).



Resolving forces radially (centripetal force) $-T + mg\cos x = -mL\,\dot{x}^2$

Resolving forces transverse to the pendulum: $-mg\sin x = mL\,\ddot{x}$

$$\Rightarrow \quad \ddot{x} + k^2 \sin x = 0\,, \qquad \text{where} \quad k^2 = \frac{g}{L} \qquad \textbf{(1)}$$

The Maclaurin series expansion of $\sin x$ is:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$$

Provided the oscillations of the pendulum are small, ($x \ll 1$), $\sin x \approx x$ and the ordinary differential equation governing the motion of the pendulum is, to a good approximation,

$$\frac{d^2 x}{dt^2} + k^2 x = 0 \qquad \textbf{(2)}$$

(which is the ODE of simple harmonic motion)

Let the angular velocity of the pendulum be $v = \dot{x} = \dfrac{dx}{dt}$.

Then, using the chain rule of differentiation,

$$\ddot{x} = \dot{v} = \frac{dv}{dt} = \frac{dv}{dx}\cdot\frac{dx}{dt} = v\frac{dv}{dx}$$

The ODE **(2)** becomes

$$v\frac{dv}{dx} + k^2 x = 0 \qquad \Rightarrow \quad v\,dv + k^2 x\,dx = 0$$

$$\Rightarrow \quad \int v\,dv + k^2 \int x\,dx = 0 \qquad \Rightarrow \quad \frac{v^2}{2} + k^2\frac{x^2}{2} = \frac{c}{2} \qquad \Rightarrow \quad v^2 + k^2x^2 = c \qquad \textbf{(3)}$$

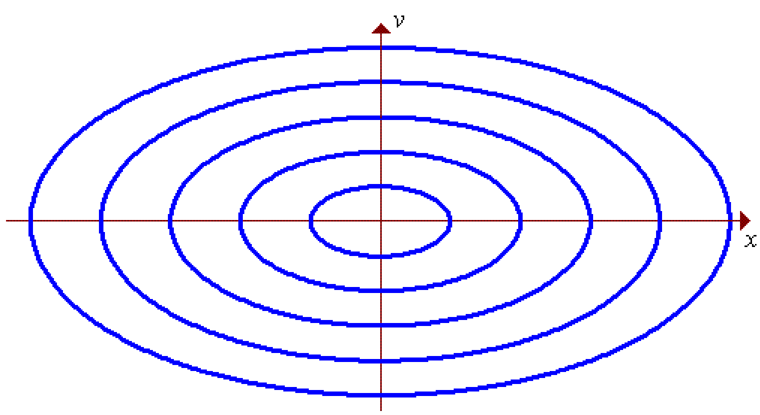
If at time $t = 0$ the pendulum is passing through its equilibrium position with angular speed v_o, then the initial conditions are

$$x(0) = 0\,, \qquad \left.\frac{dx}{dt}\right|_{t=0} = v(0) = v_o$$

Substituting the initial conditions into **(2)**, $v_o^{\,2} + k^2 \times 0 = c$, which leads to a complete solution for v as an implicit function of x,

$$v^2 + k^2x^2 = v_o^{\,2} \qquad \textbf{(4)}$$

A plot of this solution for various choices of v_o generates a family of concentric ellipses.



Recall that this solution is valid only for small displacements x.

The x-v plane is called the **phase plane**.

Returning to the more general case

$$\ddot{x} + k^2 \sin x = 0, \qquad \text{where} \quad k^2 = \frac{g}{L} \qquad \textbf{(1)}$$

and again using

$$\ddot{x} = \dot{v} = \frac{dv}{dt} = \frac{dv}{dx}\cdot\frac{dx}{dt} = v\frac{dv}{dx}$$

the ODE can be re-written as

$$v\frac{dv}{dx} + k^2 \sin x = 0 \quad \Rightarrow \quad v\,dv + k^2 \sin x\,dx = 0$$

$$\Rightarrow \int v\,dv + k^2 \int \sin x\,dx = 0 \quad \Rightarrow \quad \frac{v^2}{2} - k^2 \cos x = \frac{c}{2} \qquad \textbf{(5)}$$

$$\Rightarrow \underbrace{\frac{1}{2}mL^2v^2}_{K.E.} + \underbrace{\left(-mL^2k^2 \cos x\right)}_{P.E.} = \underbrace{\frac{1}{2}mL^2c}_{\text{Total energy}}$$

However, the kinetic energy is $\frac{1}{2}m(Lv)^2$. In the absence of friction, the sum of kinetic and potential energy is constant, so that the potential energy of the pendulum must be $-mL^2k^2 \cos x$ (= $-mg\,L \cos x$, which makes sense upon examining the diagram on page 4.02). Each value of total energy $E = \frac{1}{2}mL^2c$ generates an **orbit** (or energy curve). The relationship between total energy and initial angular velocity is obtained from substituting the initial conditions ($x = 0$ and $v = v_o$ when $t = 0$) into **(5)**:

$$\frac{v_o^{\,2}}{2} - k^2 = \frac{c}{2} \quad \Rightarrow \quad c = v_o^{\,2} - 2k^2$$

$$\Rightarrow \quad v^2 - 2k^2 \cos x = v_o^{\,2} - 2k^2 \qquad \textbf{(6)}$$

$$\Rightarrow \quad v_o^{\,2} - v^2 = 2k^2 - 2k^2 \cos x = 2k^2\left(1 - \cos x\right) \ge 0$$

$$\Rightarrow \quad v_o^{\,2} - v^2 \ge 0 \quad \Rightarrow \quad v_o^{\,2} \ge v^2 \quad \Rightarrow \quad |v| \le |v_o|$$

The maximum angular speed of the pendulum, v_o, occurs when $x = 0$.

Also, using $\cos 2\theta = 1 - 2\sin^2\theta \;\Rightarrow\; 2\sin^2\theta = 1 - \cos 2\theta$,

$$\frac{v_o^{\,2} - v^2}{2k^2} = 1 - \cos x = 2\sin^2\left(\frac{x}{2}\right)$$

$$\therefore \quad \frac{v_o^{\,2} - v^2}{4k^2} = \sin^2\left(\frac{x}{2}\right)$$

Recall that the angular velocity is just $v = \dfrac{dx}{dt}$.

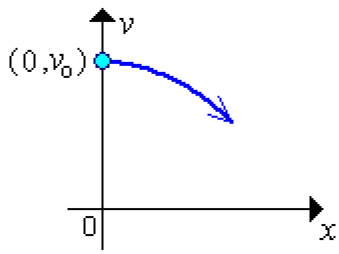
Differentiating the complete solutions **(6)**: $v^2 - 2k^2 \cos x = v_o^{\,2} - 2k^2$ implicitly with respect to time, we obtain

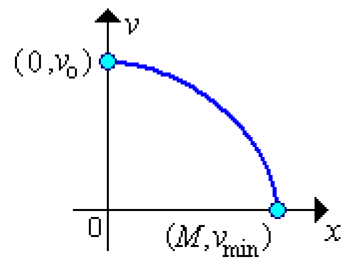
$$2v\frac{dv}{dt} + 2k^2 \sin x \frac{dx}{dt} = 0 \qquad \Rightarrow \quad \frac{dv}{dt} = -k^2 \sin x$$

This expression can also be derived directly from the ODE $\ddot{x} + k^2 \sin x = 0$ **(1)**.

When $0 \le x \le \pi$ and $v > 0$, $\dfrac{dx}{dt} = v > 0$ and $\dfrac{d^2x}{dt^2} = \dfrac{dv}{dt} = -k^2 \sin x < 0$.

Therefore, in the phase plane, as x increases from the starting point $(0, v_o)$, v decreases in the first quadrant, until the maximum value of x (label that maximum value of x as M).

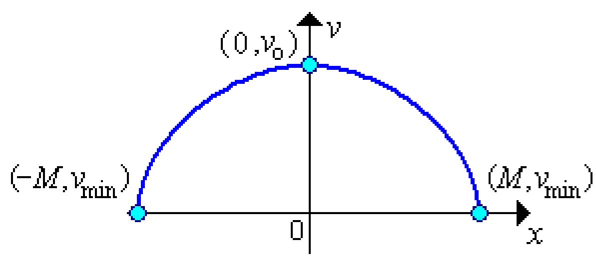




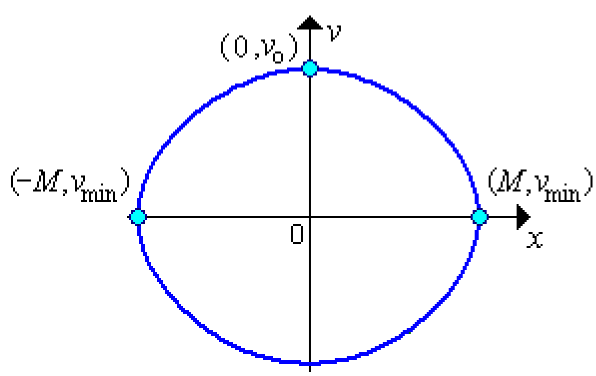
Tracking back into the second quadrant, before $(0, v_o)$,

$$-\pi \le x \le 0 \text{ and } v > 0 \Rightarrow \frac{dx}{dt} = v > 0 \text{ and } \frac{d^2x}{dt^2} = \frac{dv}{dt} = -k^2 \sin x > 0 .$$

The orbit increases from $x = -M$ to a maximum at $(0, v_o)$, then decreases until $x = +M$.
This tracks the motion of the pendulum on its complete swing from left to right.



By symmetry, the swing in the opposite direction should generate a mirror image in the x axis of the phase plane, to complete the orbit.



(6) $\Rightarrow \quad v^2 - 2k^2 \cos x = v_o^{\,2} - 2k^2$

$$\Rightarrow \quad v^2 = v_o^{\,2} - 2k^2\left(1 - \cos x\right) = v_o^{\,2} - 2k^2\left(2\sin^2\left(\frac{x}{2}\right)\right)$$

$$\Rightarrow \quad v^2 = v_o^{\,2} - \left(2k\sin\left(\frac{x}{2}\right)\right)^2 \qquad \textbf{(7)}$$

Three cases arise:

$|\, v_o \,| < 2k$:

v will decrease to zero:

$$v = 0 \quad \Rightarrow \quad 0 = v_o^{\,2} - \left(2k\sin\left(\frac{x}{2}\right)\right)^2 \quad \Rightarrow \quad \sin\left(\frac{x}{2}\right) = \pm\frac{v_o}{2k}$$

In the first quadrant of the phase plane, the orbit will move right and down to an intercept on the x axis at $(M, 0)$, where $\sin\left(\frac{M}{2}\right) = +\frac{v_o}{2k}$ and $0 < M < \pi$. Extending to the other three quadrants, the orbits resemble ellipses, centred on the origin.

$|\, v_o \,| = 2k$:

v will just barely decrease to zero:

$$v = 0 \quad \Rightarrow \quad 0 = \left(2k\right)^2 - \left(2k\sin\left(\frac{x}{2}\right)\right)^2 = \left(2k\right)^2\left(\cos\left(\frac{x}{2}\right)\right)^2 \quad \Rightarrow \quad M = \pm\pi$$

The pendulum swings all the way to the upside-down position and comes to rest there, before either swinging back or continuing on in the same direction.

$|\, v_o \,| > 2k$:

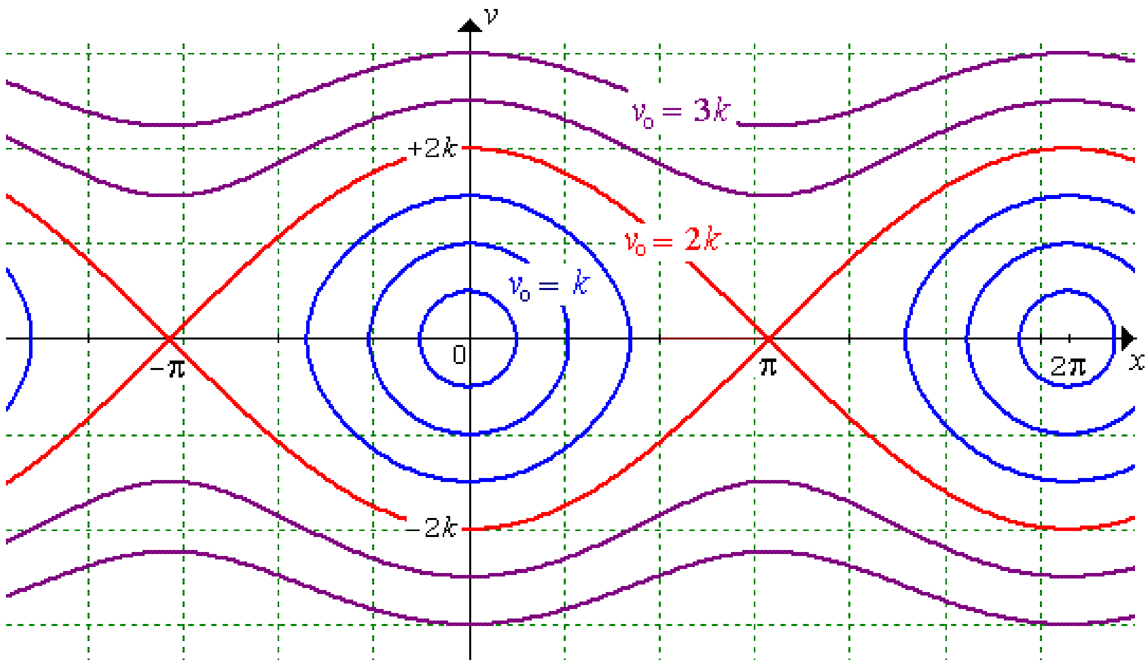
The pendulum will never come to rest, reaching a non-zero minimum speed as it passes through the upside-down position:

$$\sin\left(\frac{x}{2}\right) = \pm\frac{v_o}{2k} \quad \text{has no real solution for } x \text{ when } |\, v_o \,| > 2k.$$

If it is swinging anticlockwise, then the orbit stays in the first and second quadrants.
If it is swinging clockwise, then the orbit stays in the third and fourth quadrants.
These orbits are **not** closed and extend beyond the range $-\pi \le x \le \pi$.

$$v_{\min} = v_o^{\,2} - \left(2k\sin\left(\frac{\pi}{2}\right)\right)^2 = v_o^{\,2} - 4k^2$$

We can then generate the full set of orbits in the phase plane for the general pendulum problem.



As time progresses, one moves along an orbit to the right above the x axis, but to the left below the x axis $\left(\because \quad v = \dfrac{dx}{dt} \right)$.

The relationship **(7)** between angular velocity v and angle x is itself a first order non-linear ordinary differential equation for x as a function of the time t:

$$\left(\frac{dx}{dt} \right)^2 = v_o^{\,2} - \left(2k \sin\left(\frac{x}{2} \right) \right)^2 \quad \Rightarrow \quad \frac{dx}{dt} = \pm \sqrt{v_o^{\,2} - \left(2k \sin\left(\frac{x}{2} \right) \right)^2}$$

$$\Rightarrow \quad \pm \int \frac{dx}{\sqrt{v_o^{\,2} - 4k^2 \sin^2\left(\frac{x}{2} \right)}} = t + C$$

For the case of closed orbits ($|\, v_o \,| < 2k$), the time to complete one orbit (the period T of the pendulum) can be shown to be

$$T = \frac{4}{k} \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - b^2 \sin^2 \theta}}, \quad \text{where} \quad b = \sin \frac{M}{2} = \frac{v_o}{2k} \quad \text{and} \quad k = \sqrt{\frac{g}{L}}$$

This is a complete elliptic integral of the first kind, which has no analytic solution in terms of finite combinations of algebraic functions, (except for special choices of v_o and k). As $v_o \rightarrow 2k$, the period T diverges to infinity – it takes forever for the zero energy pendulum to reach the upside-down position.

4.02 Stability of Stationary Points

Consider the (generally non-linear) system of simultaneous first order ordinary differential equations

$$\frac{dx}{dt} = \dot{x} = P(x,y), \qquad \frac{dy}{dt} = \dot{y} = Q(x,y) \qquad \textbf{(1)}$$

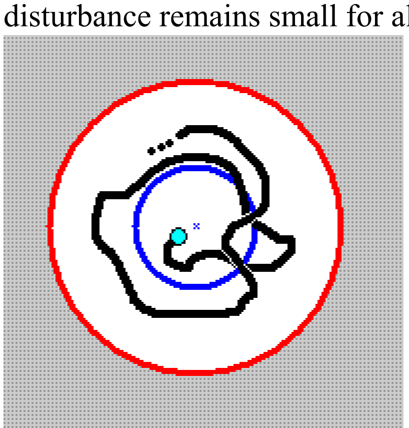
Using the chain rule, $\dfrac{dy}{dx} = \dfrac{dy}{dt} \cdot \dfrac{dt}{dx} = \dfrac{\dot{y}}{\dot{x}} = \dfrac{Q(x,y)}{P(x,y)}$.

This can be integrated with respect to x to obtain a solution for y as an implicit function of x, provided $P(x, y) \neq 0$. At points where $P(x, y) = 0$ but $Q(x, y) \neq 0$, one may integrate $\dfrac{dx}{dy} = \dfrac{P(x,y)}{Q(x,y)}$ instead.

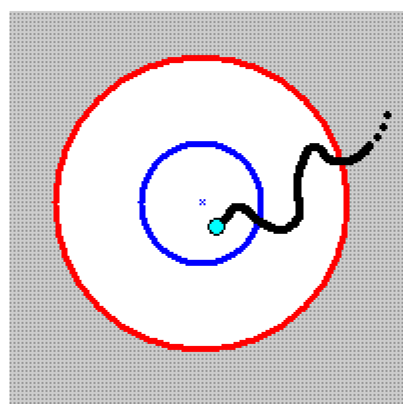
Points on the phase plane where $P(x, y) = Q(x, y) = 0$ are **singular points**.
A unique slope does not exist at such points.

Alternative names for singular points are **equilibrium** points or **stationary** points (because both x and y do not [instantaneously] change with time there) or **critical** points or **fixed** points.

A singular point is **stable** (and is called an "**attractor**") if the response to a small disturbance remains small for all time.



Stable singular point:
all paths starting inside the inner circle stay closer than the outer circle forever.



Unstable singular point.
(or "source")

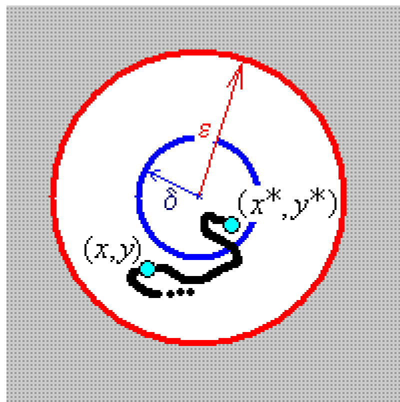
Consider the system

$$\dot{x} = P(x,y), \quad \dot{y} = Q(x,y), \quad x(0) = x^*, \quad y(0) = y^*, \quad P(0,0) = Q(0,0) = 0 \qquad \textbf{(2)}$$

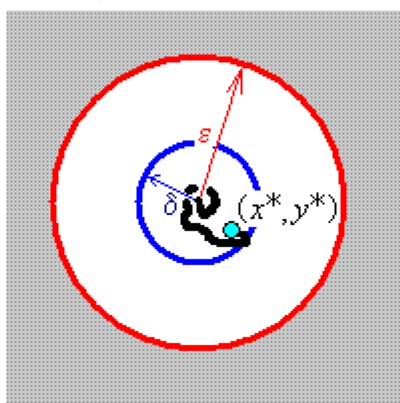
which has a stationary point at the origin.
Let $x(t; x^*)$, $y(t; y^*)$ be the complete solution to this system.

The stationary point at the origin is stable if and only if,
for every $\varepsilon > 0$ (however small), there exists a $\delta(\varepsilon)$ such that whenever the point $(x^*, y^*) = (x(0; x^*), y(0; y^*))$ is closer than δ to the origin, the point $(x(t; x^*), y(t; y^*))$ remains closer than ε to the origin for all time, or

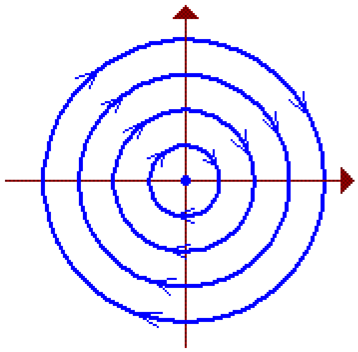
$$\sqrt{(x^*)^2 + (y^*)^2} < \delta \quad \Rightarrow \quad \sqrt{x^2(t;x^*) + y^2(t;y^*)} < \varepsilon \quad \forall t$$



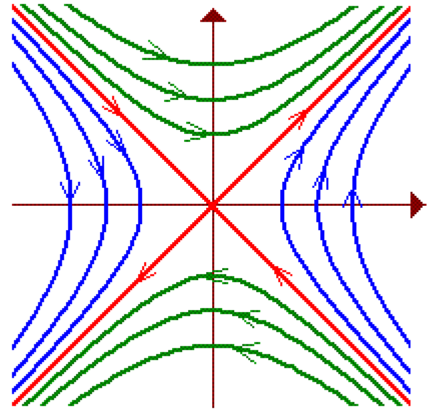
A stationary point is **asymptotically stable** (also known as a "sink") if it is stable and any disturbance ultimately vanishes: $\lim_{t\to\infty} \left[x^2(t;x^*) + y^2(t;y^*) \right] = 0$.



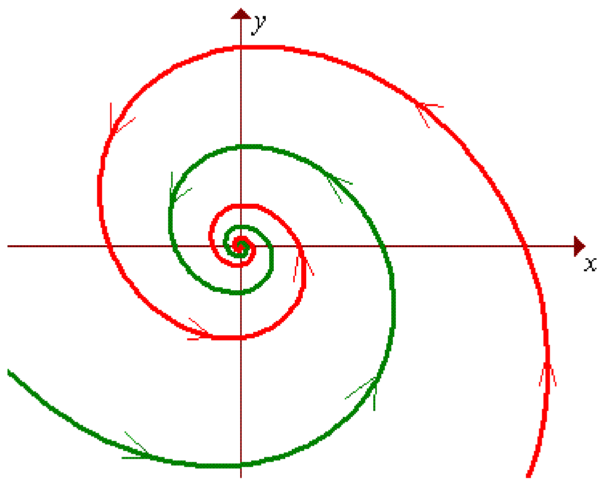
Here are three types of stationary points with nearby orbits:



Stable centre
(but *not* asymptotically stable)



Unstable saddle point
[all saddle points are unstable]



Asymptotically stable focus
(or spiral sink)

4.03 Linear Approximation to a System of Non-Linear ODEs (1)

The Taylor series of any function $f(x, y)$ about the point (x_0, y_0) is

$$f(x,y) = f(x_0,y_0) + \left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} \cdot (x-x_0) + \left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} \cdot (y-y_0) + \left.\frac{\partial^2 f}{\partial x^2}\right|_{(x_0,y_0)} \frac{(x-x_0)^2}{2!} + 2\left.\frac{\partial^2 f}{\partial x\,\partial y}\right|_{(x_0,y_0)} \frac{(x-x_0)(y-y_0)}{2!} + \left.\frac{\partial^2 f}{\partial y^2}\right|_{(x_0,y_0)} \frac{(y-y_0)^2}{2!} + \dots \quad \textbf{(1)}$$

provided that the series converges to $f(x, y)$.

This allows us to create a linear approximation to the non-linear system

$$\dot{x} = P(x,y), \quad \dot{y} = Q(x,y), \quad x(0) = x^*, \quad y(0) = y^*, \quad P(0,0) = Q(0,0) = 0. \quad \textbf{(2)}$$

$$P(x,y) = P(0,0) + \left.\frac{\partial P}{\partial x}\right|_{(0,0)} x + \left.\frac{\partial P}{\partial y}\right|_{(0,0)} y + P_1(x,y) \quad \textbf{(3)}$$

where $\displaystyle\lim_{(x,y)\to(0,0)} \frac{P_1(x,y)}{\sqrt{x^2+y^2}} = 0$, (because $P_1(x, y)$ is at least second order in x, y)

and similarly for $Q(x, y)$, so that the system becomes

$$\begin{aligned} \dot{x} &= ax + by + P_1(x,y) \\ \dot{y} &= cx + dy + Q_1(x,y) \end{aligned} \quad \textbf{(4)}$$

where a, b, c, d are all constants.

In the neighbourhood of the singular point (0, 0), this system can be modelled by the linear system

$$\begin{aligned} \dot{x} &= ax + by \\ \dot{y} &= cx + dy \end{aligned} \quad \textbf{(5)}$$

where $a = \left.\frac{\partial P}{\partial x}\right|_{(0,0)}, \quad b = \left.\frac{\partial P}{\partial y}\right|_{(0,0)}, \quad c = \left.\frac{\partial Q}{\partial x}\right|_{(0,0)}, \quad d = \left.\frac{\partial Q}{\partial y}\right|_{(0,0)}.$

4.04 Reminder of Linear Ordinary Differential Equations

To find the general solution of the homogeneous second order linear ODE

$$\frac{d^2 y}{dx^2} + p\frac{dy}{dx} + q\,y = 0\,,$$

with constant real coefficients p and q,
form the **auxiliary equation** or **characteristic equation**

$$\lambda^2 + p\,\lambda + q = 0$$

and evaluate the discriminant $D = p^2 - 4q$ and the roots $\lambda_1, \lambda_2 = \frac{-p \pm \sqrt{D}}{2}$.

Three cases arise.

$D > 0$: The characteristic equation has a pair of distinct real roots λ_1, λ_2.
The general solution is $y = A e^{\lambda_1 x} + B e^{\lambda_2 x}$.

$D = 0$: The characteristic equation has a pair of equal real roots λ.
The general solution is $y = (Ax + B)e^{\lambda x}$.

$D < 0$: The characteristic equation has a complex conjugate pair of roots $\lambda_1, \lambda_2 = a \pm bj$.
The general solution is $y = e^{ax}\left(A\cos bx + B\sin bx\right)$,

where A, B are arbitrary constants of integration.

To find the general solution of the system of simultaneous first order linear ODEs

$$\frac{dx}{dt} = ax + by$$

$$\frac{dy}{dt} = cx + dy$$

substitute the trial solution $\left(x(t), y(t)\right) = \left(\alpha e^{\lambda t}, \beta e^{\lambda t}\right)$ into the ODE, to obtain

$$\alpha \lambda \cancel{e^{\lambda t}} = a\alpha \cancel{e^{\lambda t}} + b\beta \cancel{e^{\lambda t}}$$

$$\beta \lambda \cancel{e^{\lambda t}} = c\alpha \cancel{e^{\lambda t}} + d\beta \cancel{e^{\lambda t}}$$

or, in matrix form,

$$\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\alpha = \beta = 0$ is a solution (the trivial solution) for any choice of a, b, c, d and λ.
Non-trivial solutions exist when the determinant of the matrix of coefficients is zero:

$$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0 \quad \Rightarrow \quad (a-\lambda)(d-\lambda) - bc = 0$$

$$\Rightarrow \quad \lambda^2 - (a+d)\lambda + (ad-bc) = 0$$

which is the **characteristic equation** of the system.

The solutions to the characteristic equation are the **eigenvalues** of the coefficient matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and, for each eigenvalue λ, a non-zero vector $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ that satisfies the equation

$$\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

is an **eigenvector** for that eigenvalue.

The general solution to the system of ODEs is a linear combination of the solutions arising from each eigenvalue:

$$\left(x(t), y(t)\right) = \left(c_1\alpha_1 e^{\lambda_1 t} + c_2\alpha_2 e^{\lambda_2 t},\ c_1\beta_1 e^{\lambda_1 t} + c_2\beta_2 e^{\lambda_2 t}\right)$$

unless the eigenvalues are equal, in which case the general solution is

$$\left(x(t), y(t)\right) = \left(\left(c_1\alpha_1 + c_2\alpha_2 t\right)e^{\lambda t},\ \left(c_1\beta_1 + c_2\beta_2 t\right)e^{\lambda t}\right)$$

(where, in this case, (α_1, β_1) is not necessarily an eigenvector).

4.05 Stability Analysis for a Linear System

In the case where (0, 0) is the only critical point of the system

$$\frac{dx}{dt} = ax + by$$
$$\frac{dy}{dt} = cx + dy$$

it follows that the characteristic equation $\lambda^2 - (a+d)\lambda + (ad-bc) = 0$
has only non-zero roots and that $\det A = ad - bc \neq 0$.

<u>Proof</u>:

If $\lambda = 0$ then at least one eigenvalue of the coefficient matrix A is zero, from which it follows immediately that

$$\begin{vmatrix} a-0 & b \\ c & d-0 \end{vmatrix} = \det A = ad - bc = 0$$

Both roots non-zero $\Rightarrow$ $ad - bc \neq 0$.

If (0, 0) is the only critical point of the system, then no other choice of (x, y) satisfies both equations

$$\begin{array}{c} ax + by = 0 \\ cx + dy = 0 \end{array} \quad \Rightarrow \quad \begin{array}{c} (ad-bc)x = 0 \\ (ad-bc)y = 0 \end{array}$$

from which it follows immediately that $ad - bc = \det A \neq 0$.

If the roots are both non-zero and (x, y) is a critical point of the system, then

$$\begin{array}{c} ax + by = 0 \\ cx + dy = 0 \end{array} \quad \Rightarrow \quad \begin{array}{c} (ad-bc)x = 0 \\ (ad-bc)y = 0 \end{array}$$

But $\lambda_1, \lambda_2 \neq 0 \Rightarrow ad - bc \neq 0 \Rightarrow$ (0, 0) is the only solution to this pair of simultaneous linear equations.

Therefore (0, 0) is the only critical point of the system if and only if both roots of the characteristic equation are non-zero.