4.11 Duffing's Equation

Among the simplest models of damped non-linear forced oscillations of a mechanical or electrical system with a cubic stiffness term is Duffing's equation:

$$\frac{d^2 x}{dt^2} + a\frac{dx}{dt} + bx + cx^3 = d\cos\omega t \qquad \textbf{(1)}$$

In section 4.01, we considered the simple undamped pendulum:

$$\frac{d^2 x}{dt^2} + \frac{g}{L}\sin x = 0 \qquad \textbf{(2)}$$

When x is very small, $\sin x \approx x$ and **(2)** reduces to the ODE for simple harmonic motion.

The next order approximation is $\sin x \approx x - \frac{x^3}{6}$, so that **(2)** becomes

$$\frac{d^2 x}{dt^2} + \frac{g}{L}x - \frac{g}{L}\frac{x^3}{6} = 0 \qquad \textbf{(3)}$$

If we add a damping term $a\frac{dx}{dt}$ and a forcing function $d\cos\omega t$, then **(3)** becomes Duffing's equation **(1)**.

Special Case 1:

Conduct a stability analysis for the undamped unforced Duffing's equation

$$\frac{d^2x}{dt^2} + \omega^2 x + c\,x^3 = 0 \qquad \textbf{(4)}$$

The equivalent first order system is

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = -\omega^2 x - c\,x^3 \qquad \textbf{(5)}$$

Critical points:

$$\left(y=0\right) \quad \text{and} \quad \left(x=0 \quad \text{or} \quad x^2 = -\frac{\omega^2}{c}\right)$$

Near (0, 0) the linear approximation is

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \qquad \textbf{(6)}$$

The characteristic equation is $\det(A - \lambda I) = 0 \quad \Rightarrow \quad \lambda^2 + \omega^2 = 0$

The eigenvalues are a pure imaginary pair

$\Rightarrow$ (0, 0) is a **centre**. It is stable but not asymptotically stable.

If $c \geq 0$, then this is the only critical point of **(4)**.

If $c < 0$, then there are two other critical points, at $\left(\pm\sqrt{\frac{\omega^2}{-c}}, 0\right)$.

Special Case 1: (continued)

Near $\left(\pm\sqrt{\frac{\omega^2}{-c}}, 0 \right)$

$$\left.\frac{dx}{dt}\right|_{\text{near}\left(\pm\omega/\sqrt{-c},0\right)} = \left.(y)\right|_{\text{near}\left(\pm\omega/\sqrt{-c},0\right)} \approx \left.\frac{\partial P}{\partial x}\right|_{\left(\pm\omega/\sqrt{-c},0\right)}\left(x \mp \sqrt{\frac{\omega^2}{-c}} \right) + \left.\frac{\partial P}{\partial y}\right|_{\left(\pm\omega/\sqrt{-c},0\right)} y$$

$$= (0)\left(x \mp \sqrt{\frac{\omega^2}{-c}} \right) + (1)\,y = y$$

$$\left.\frac{dy}{dt}\right|_{\text{near}\left(\pm\omega/\sqrt{-c},0\right)} = \left.\left(-\omega^2 x - c x^3\right)\right|_{\text{near}\left(\pm\omega/\sqrt{-c},0\right)} \approx \left.\frac{\partial Q}{\partial x}\right|_{\left(\pm\omega/\sqrt{-c},0\right)}\left(x \mp \sqrt{\frac{\omega^2}{-c}} \right) + \left.\frac{\partial Q}{\partial y}\right|_{\left(\pm\omega/\sqrt{-c},0\right)} y$$

$$= \left(-\omega^2 - 3c\left(\frac{\omega^2}{-c}\right) \right)\left(x \mp \sqrt{\frac{\omega^2}{-c}} \right) + (0)\,y = 2\omega^2\left(x \mp \sqrt{\frac{\omega^2}{-c}} \right)$$

The linear approximation to **(5)** near $\left(\pm\sqrt{\frac{\omega^2}{-c}}, 0 \right)$ is

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2\omega^2 & 0 \end{pmatrix}\begin{pmatrix} x \mp \sqrt{\frac{\omega^2}{-c}} \\ y \end{pmatrix} \qquad \textbf{(7)}$$

The characteristic equation is $\det(A - \lambda I) = 0 \;\Rightarrow\; \lambda^2 - 2\omega^2 = 0$
The eigenvalues are a distinct real pair with opposite sign

$\Rightarrow$ $\left(\pm\sqrt{\frac{\omega^2}{-c}}, 0 \right)$ are **saddle points**. They are unstable.

Exact Solution of Special Case 1:

The system **(5),**

$$\frac{dx}{dt} = y$$

$$\frac{dy}{dt} = -\omega^2 x - c\,x^3$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{-\omega^2 x - c\,x^3}{y}$$

$$\Rightarrow \quad y\,dy = \left(-\omega^2 x - c\,x^3\right)dx \qquad \Rightarrow \quad \frac{y^2}{2} = \frac{A}{2} - \frac{\omega^2 x^2}{2} - \frac{c\,x^4}{4}$$

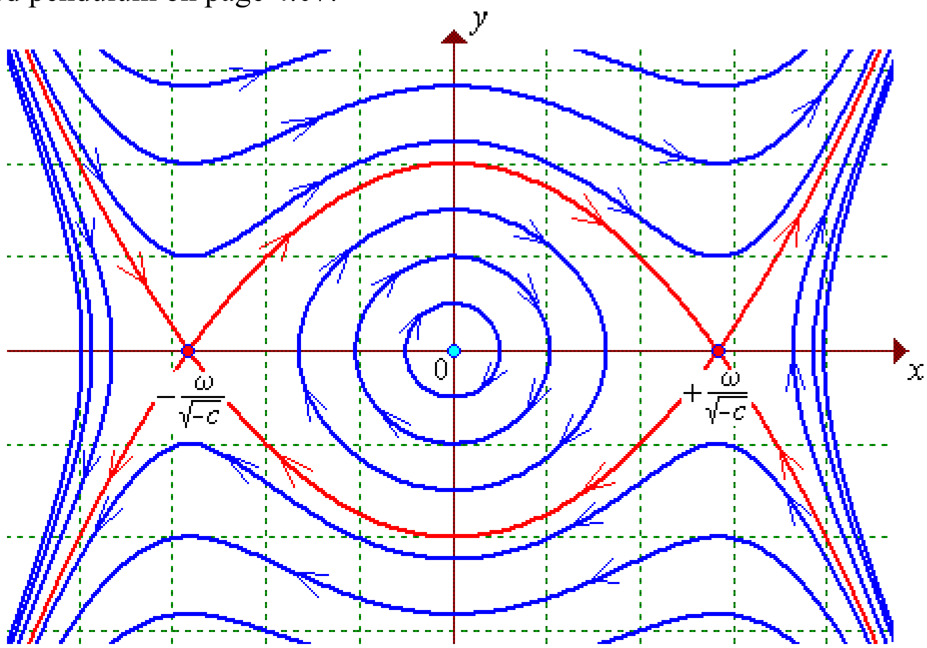
Therefore the orbits in the phase space (x, y) are $y^2 = A - \omega^2 x^2 - \frac{c\,x^4}{2}$,
where A is an arbitrary constant.

If $c \geq 0$, then all orbits are closed, about the centre at (0, 0).

Special Case 1: (continued)

If $c < 0$, then those orbits far enough away from the centre are open, due to the influence of the saddle points at $\left(\pm\sqrt{\dfrac{\omega^2}{-c}}, 0 \right)$.

The part of the phase space between the two saddle points resembles that for the undamped pendulum on page 4.07:



The orbits passing through the saddle points separates closed orbits from open orbits and is called the **separatrix**.

The positive y axis intercept of each orbit is just the value of $\sqrt{A}$ for that orbit.
The separatrix has x axis intercepts at the saddle points. Therefore, for the separatrix,

$$A = \left(\omega^2 x^2 + \frac{cx^4}{2} \right) \bigg|_{x = \pm\omega/\sqrt{-c}} = \omega^2 \left(\frac{\omega^2}{-c} \right) + \frac{c}{2} \left(\frac{\omega^4}{c^2} \right) = \frac{\omega^4}{-2c}$$

The equation of the separatrix is

$$y^2 = \frac{\omega^4}{2|c|} - \omega^2 x^2 + \frac{|c| x^4}{2}, \quad (c < 0)$$

Special Case 2:

Conduct a stability analysis for the damped unforced Duffing's equation

$$\frac{d^2x}{dt^2} + a\frac{dx}{dt} + \omega^2 x + c\,x^3 = 0 \qquad \textbf{(8)}$$

The equivalent first order system is

$$\begin{aligned} \frac{dx}{dt} &= y \\ \frac{dy}{dt} &= -\omega^2 x - c\,x^3 - a\,y \end{aligned} \qquad \textbf{(9)}$$

The critical points are the same as in special case 1:

$(y = 0)$ and $\left(x = 0 \quad \text{or} \quad x^2 = -\dfrac{\omega^2}{c}\right)$

Near (0, 0) the linear approximation is

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & -a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \qquad \textbf{(10)}$$

The characteristic equation is $\det(A - \lambda I) = 0 \quad \Rightarrow \quad \lambda^2 + a\lambda + \omega^2 = 0$

$$\Rightarrow \quad \lambda = \frac{-a \pm \sqrt{a^2 - 4\omega^2}}{2}$$

The critical point is stable if $a > 0$ and unstable if $a < 0$.
It is a focus if $a^2 - 4\omega^2 < 0$ and a node otherwise.

If $c \geq 0$, then this is the only critical point of **(8)**.

If $c < 0$, then there are two other critical points, at $\left(\pm\sqrt{\dfrac{\omega^2}{-c}}, 0\right)$.

Special Case 2: (continued)

Near $\left(\pm\sqrt{\frac{\omega^2}{-c}},\, 0 \right)$

$$\left.\frac{dx}{dt}\right|_{\text{near}\left(\pm\omega/\sqrt{-c},0\right)} = \left(y\right)\Big|_{\text{near}\left(\pm\omega/\sqrt{-c},0\right)} \approx \left.\frac{\partial P}{\partial x}\right|_{\left(\pm\omega/\sqrt{-c},0\right)} \left(x \mp \frac{\omega}{\sqrt{-c}} \right) + \left.\frac{\partial P}{\partial y}\right|_{\left(\pm\omega/\sqrt{-c},0\right)} y$$

$$= (0)\left(x \mp \frac{\omega}{\sqrt{-c}} \right) + (1)\,y = y$$

$$\left.\frac{dy}{dt}\right|_{\text{near}\left(\pm\omega/\sqrt{-c},0\right)} = \left(-\omega^2 x - c x^3 - a y\right)\Big|_{\text{near}\left(\pm\omega/\sqrt{-c},0\right)} \approx \left.\frac{\partial Q}{\partial x}\right|_{\left(\pm\omega/\sqrt{-c},0\right)} \left(x \mp \frac{\omega}{\sqrt{-c}} \right) + \left.\frac{\partial Q}{\partial y}\right|_{\left(\pm\omega/\sqrt{-c},0\right)} y$$

$$= \left(-\omega^2 - 3c\left(\frac{\omega^2}{-c} \right) \right)\left(x \mp \frac{\omega}{\sqrt{-c}} \right) + (-a)\,y = 2\omega^2 \left(x \mp \frac{\omega}{\sqrt{-c}} \right) - a\,y$$

The linear approximation to **(9)** near $\left(\pm\sqrt{\frac{\omega^2}{-c}},\, 0 \right)$ is

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2\omega^2 & -a \end{pmatrix} \begin{pmatrix} x \mp \frac{\omega}{\sqrt{-c}} \\ y \end{pmatrix} \qquad \textbf{(11)}$$

The characteristic equation is $\det(A - \lambda I) = 0 \;\Rightarrow\; \lambda^2 + a\lambda - 2\omega^2 = 0$

$$\Rightarrow \quad \lambda = \frac{-a \pm \sqrt{a^2 + 8\omega^2}}{2}$$

The eigenvalues are a distinct real pair with opposite sign

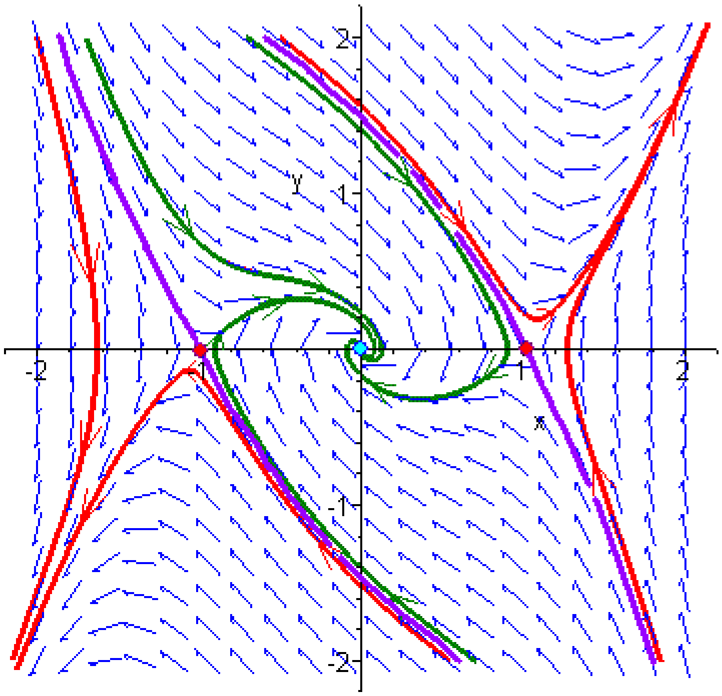
$\Rightarrow \left(\pm\sqrt{\frac{\omega^2}{-c}},\, 0 \right)$ are **saddle points**. They are unstable.

The presence of the damping term changes the centre into a stable focus (for physically reasonable values of a, ω and c, or, for particularly strong damping, a stable node). The form of the separatrix is more complicated, as trajectories leaving either saddle point in the direction of the origin are swept by the damping term into the focus (or node) instead of moving around the centre to the other saddle point. There are no closed orbits; just orbits that terminate at the origin or a saddle point and orbits that retreat to infinity.

Special Case 2: (continued)

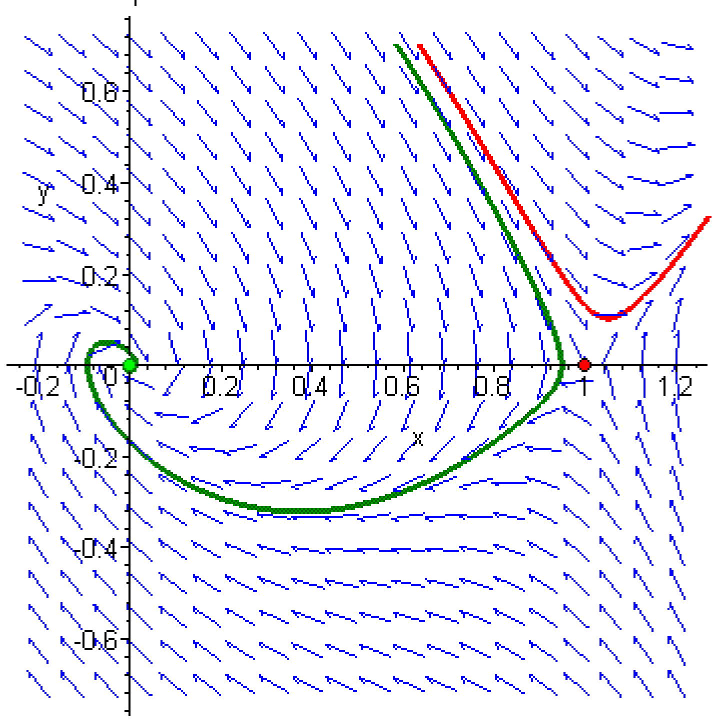
An enhanced sample phase portrait plot from Maple is shown here:

Example 4.11.1 Non-Linear Solution w = -c = a = 1



and, zooming in,

Example 4.11.1 Non-Linear Solution w = -c = a = 1



4.12 More Examples

Example 4.12.1

Examine the stability of the linear second order differential equation

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + \left(4\pi^2+1\right)x = 0$$

and find the complete solution for the initial conditions
$x(0) = 0, \quad y(0) = \dot{x}(0) = 2\pi$.

The system can be rewritten as the first order system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\left(4\pi^2+1\right) & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

The only critical point is at the origin.

$$D = (a-d)^2 + 4bc = (0+2)^2 + 4(1)\left(-\left(4\pi^2+1\right)\right) = -16\pi^2$$

$D < 0$ and $(a + d) < 0$ $\Rightarrow$ the critical point is an **asymptotically stable focus**.

$$\lambda = \frac{(a+d)\pm\sqrt{D}}{2} = \frac{-2\pm\sqrt{-16\pi^2}}{2} = -1\pm 2\pi j$$

Using the formula on page 4.30, $u=-1$, $v=2\pi$, $u-d=-1+2=1$, $c=-\left(4\pi^2+1\right)$.
The general solution is

$$x(t) = e^{-t}\left(c_3\left(\cos 2\pi t - 2\pi\sin 2\pi t\right) + c_4\left(2\pi\cos 2\pi t + \sin 2\pi t\right)\right)$$

$$y(t) = -e^{-t}\left(4\pi^2+1\right)\left(c_3\cos 2\pi t + c_4\sin 2\pi t\right)$$

[and one can check that $\frac{dx}{dt} = y$ is indeed true.]

$$\left(x(0), y(0)\right) = (0, 2\pi) \quad \Rightarrow \quad \left(c_3 + 2\pi c_4,\ -c_3\right) = \left(0, \frac{2\pi}{4\pi^2+1}\right)$$

$$\Rightarrow \quad \left(c_3,\ c_4\right) = \left(\frac{-2\pi}{4\pi^2+1}, \frac{1}{4\pi^2+1}\right)$$

Example 4.12.1 (continued)

The complete solution is

$$\left(x(t), y(t)\right) = \frac{e^{-t}}{4\pi^2+1}\times$$

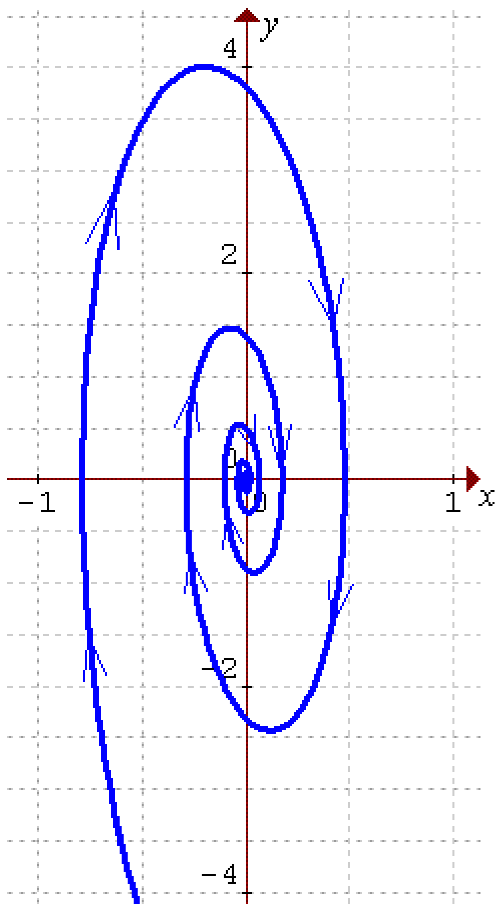
$$\left(2\pi\left(2\pi\sin 2\pi t - \cos 2\pi t\right) + \left(2\pi\cos 2\pi t + \sin 2\pi t\right),\ \left(4\pi^2+1\right)\left(2\pi\cos 2\pi t\ -\ \sin 2\pi t\right)\right)$$

$\Rightarrow$

$$\boxed{\left(x(t), y(t)\right) = e^{-t}\left(\sin 2\pi t,\ \left(2\pi\cos 2\pi t\ -\ \sin 2\pi t\right)\right)}$$

As $t \to \infty$, both functions $x(t)$ and $y(t)$ tend to zero.

The resulting phase space diagram is



Example 4.12.2

Examine the stability of the linear second order differential equation

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + \left(4\pi^2+1\right)x = \left(3\pi^2+1\right)\cos\pi t - 2\pi\sin\pi t$$

The complete solution for the initial conditions

$x(0) = 1, \quad y(0) = \dot{x}(0) = 2\pi$.

can be obtained by building upon the solution to Example 4.12.1 and is

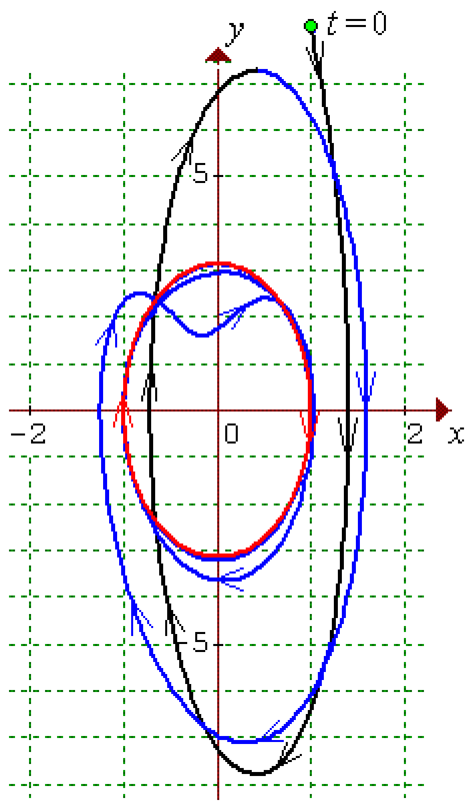
$$\boxed{\left(x(t), y(t)\right) = \left(\cos\pi t + e^{-t}\sin 2\pi t, \; -\pi\sin\pi t + e^{-t}\left(2\pi\cos 2\pi t - \sin 2\pi t\right)\right)}$$

In this case, the steady state solution (after the transient terms have vanished) is

$\lim_{t\to\infty}\left(x(t), y(t)\right) = \left(\cos\pi t, \; -\pi\sin\pi t\right)$

so that the orbit in the phase space approaches the ellipse

$$\pi^2x^2 + y^2 = \pi^2$$



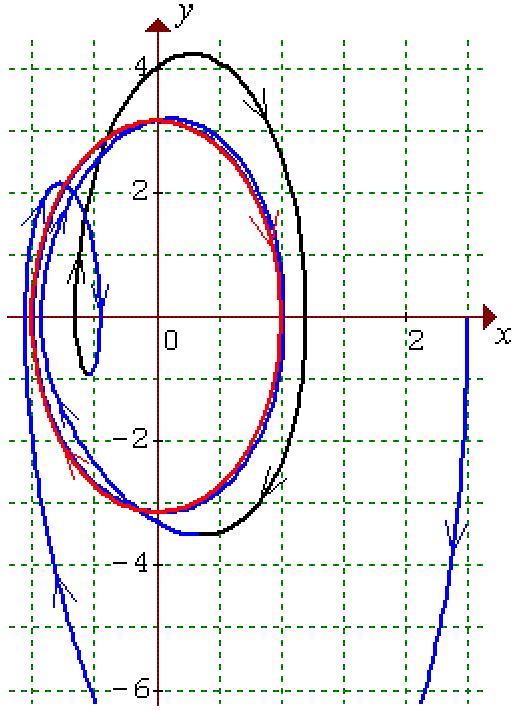
This ellipse must therefore be the limit cycle for the system.

A plot of the orbit in the phase space is shown here.

Note how the solution curve in the phase space can wander inside and outside the limit cycle more than once, before finally settling down to its asymptotic approach as the transient terms become negligible.

Example 4.12.2 (continued)

Different sets of initial conditions can generate orbits that look very different at first, before they settle down into their steady-state configuration near the limit cycle.



4.13 Liénard’s Theorem

If $f(x)$ is an even function for all x
and $g(x)$ is an odd function for all x
and $g(x) > 0$ for all $x > 0$
and $F(x) = \int_0^x f(t)\,dt$ is such that $F(x) = 0$ has exactly one positive root, γ, and
$F(x) < 0$ for $0 < x < \gamma$ and $F(x) > 0$ and non-decreasing for $x > \gamma$,
then

the system

$$\dot{x} = y\ ,\quad \dot{y} = -f(x)\,y - g(x)$$

or, equivalently,

$$\frac{d^2x}{dt^2} + f(x)\frac{dx}{dt} + g(x) = 0$$

has a unique limit cycle enclosing the origin and that limit cycle is asymptotically stable.

When all of the conditions of Liénard’s theorem are satisfied, the system has exactly one periodic solution, towards which all other trajectories spiral as $t \to \infty$.

Example 4.13.1

Let $f(x) = -\mu(1 - x^2)$ and $g(x) = x$, (with $\mu > 0$),
then Liénard’s ODE becomes

$$\frac{d^2x}{dt^2} - \mu\left(1 - x^2\right)\frac{dx}{dt} + x = 0$$

which is Van der Pol’s equation (section 4.08).

Checking the conditions of Liénard’s theorem:
$f(x) = -\mu(1 - x^2)$ is an even function.
$g(x) = x$ is an odd function, positive for all $x > 0$.

$$F(x) = \int_0^x -\mu\left(1 - t^2\right)dt = -\mu\left[\left(t - \frac{t^3}{3}\right)\right]_0^x = +\mu\left(\frac{x^3}{3} - x\right) = \mu x\left(\frac{x^2}{3} - 1\right)$$

$F(x) = 0$ has only one positive root, $\gamma = \sqrt{3}$.
$F(x) < 0$ for $0 < x < \sqrt{3}$ and $F(x) > 0$ and increasing for $x > \sqrt{3}$.

Therefore Van der Pol’s equation possesses a unique and asymptotically stable limit cycle.

5. The Gradient Operator

A brief review is provided here for the gradient operator $\vec{\nabla}$ in both Cartesian and orthogonal non-Cartesian coordinate systems.

Sections in this Chapter:

5.01 Gradient, Divergence, Curl and Laplacian (Cartesian)
5.02 Differentiation in Orthogonal Curvilinear Coordinate Systems
5.03 Summary Table for the Gradient Operator
5.04 Derivatives of Basis Vectors

5.01 Gradient, Divergence, Curl and Laplacian (Cartesian)

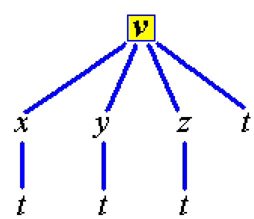
Let z be a function of two independent variables (x, y), so that $z = f(x, y)$.
The function $z = f(x, y)$ defines a surface in $\mathbb{R}^3$.
At any point (x, y) in the x-y plane, the direction in which one must travel in order to experience the greatest possible rate of increase in z at that point is the direction of the **gradient vector**,

$$\vec{\nabla} f = \frac{\partial f}{\partial x}\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\hat{\mathbf{j}}$$

The magnitude of the gradient vector is that greatest possible rate of increase in z at that point. The gradient vector is not constant everywhere, unless the surface is a plane. (The symbol $\vec{\nabla}$ is usually pronounced "del").

The concept of the gradient vector can be extended to functions of any number of variables. If $u = f(x, y, z, t)$, then $\vec{\nabla} f = \left[\frac{\partial f}{\partial x} \ \frac{\partial f}{\partial y} \ \frac{\partial f}{\partial z} \ \frac{\partial f}{\partial t}\right]^{\mathrm{T}}$.

If $\mathbf{v}$ is a function of position $\mathbf{r}$ and time t, while position is in turn a function of time, then by the chain rule of differentiation,



$$\frac{d\vec{\mathbf{v}}}{dt} = \frac{\partial \vec{\mathbf{v}}}{\partial x}\frac{dx}{dt} + \frac{\partial \vec{\mathbf{v}}}{\partial y}\frac{dy}{dt} + \frac{\partial \vec{\mathbf{v}}}{\partial z}\frac{dz}{dt} + \frac{\partial \vec{\mathbf{v}}}{\partial t} = \left(\frac{d\vec{\mathbf{r}}}{dt}\bullet\vec{\nabla}\right)\vec{\mathbf{v}} + \frac{\partial \vec{\mathbf{v}}}{\partial t}$$

$$\Rightarrow \frac{d\vec{\mathbf{v}}}{dt} = \left(\vec{\mathbf{v}}\bullet\vec{\nabla}\right)\vec{\mathbf{v}} + \frac{\partial \vec{\mathbf{v}}}{\partial t}$$

which is of use in the study of fluid dynamics.

The gradient operator can also be applied to vectors via the scalar (dot) and vector (cross) products:

The **divergence** of a vector field **F**(x, y, z) is

$$\text{div}\,\vec{\mathbf{F}} = \vec{\nabla}\bullet\vec{\mathbf{F}} = \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{bmatrix}^{\mathrm{T}} \bullet \begin{bmatrix} F_1 & F_2 & F_3 \end{bmatrix}^{\mathrm{T}} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

A region free of sources and sinks will have zero divergence:
the total flux into any region is balanced by the total flux out from that region.

The **curl** of a vector field **F**(x, y, z) is

$$\text{curl}\,\vec{\mathbf{F}} = \vec{\nabla}\times\vec{\mathbf{F}} = \begin{vmatrix} \hat{\mathbf{i}} & \frac{\partial}{\partial x} & F_1 \\ \hat{\mathbf{j}} & \frac{\partial}{\partial y} & F_2 \\ \hat{\mathbf{k}} & \frac{\partial}{\partial z} & F_3 \end{vmatrix} = \begin{bmatrix} \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \\ \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \\ \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \end{bmatrix}$$

In an irrotational field, curl $\vec{\mathbf{F}} = \vec{\mathbf{0}}$.
Whenever $\vec{\mathbf{F}} = \vec{\nabla}\phi$ for some twice differentiable potential function ϕ, curl $\vec{\mathbf{F}} = \vec{\mathbf{0}}$
or

$$\text{curl}\left(\text{grad}\,\phi\right) \equiv \vec{\nabla}\times\vec{\nabla}\phi \equiv \vec{\mathbf{0}}$$

Proof:

$$\vec{\mathbf{F}} = \vec{\nabla}\phi = \begin{bmatrix} F_1 & F_2 & F_3 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{bmatrix}^{\mathrm{T}}$$

$$\Rightarrow \quad \text{curl}\,\vec{\nabla}\phi = \begin{vmatrix} \hat{\mathbf{i}} & \frac{\partial}{\partial x} & \frac{\partial\phi}{\partial x} \\ \hat{\mathbf{j}} & \frac{\partial}{\partial y} & \frac{\partial\phi}{\partial y} \\ \hat{\mathbf{k}} & \frac{\partial}{\partial z} & \frac{\partial\phi}{\partial z} \end{vmatrix} = \begin{bmatrix} \frac{\partial^2\phi}{\partial y\,\partial z} - \frac{\partial^2\phi}{\partial z\,\partial y} \\ \frac{\partial^2\phi}{\partial z\,\partial x} - \frac{\partial^2\phi}{\partial x\,\partial z} \\ \frac{\partial^2\phi}{\partial x\,\partial y} - \frac{\partial^2\phi}{\partial y\,\partial x} \end{bmatrix} \equiv \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Among many identities involving the gradient operator is

$$\operatorname{div}\left(\operatorname{curl}\vec{\mathbf{F}}\right) \equiv \vec{\boldsymbol{\nabla}}\bullet\vec{\boldsymbol{\nabla}}\times\vec{\mathbf{F}} \equiv 0$$

for all twice-differentiable vector functions $\vec{\mathbf{F}}$

<u>Proof</u>:

$$\operatorname{div}\operatorname{curl}\vec{\mathbf{F}} = \frac{\partial}{\partial x}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

$$= \frac{\partial^2 F_3}{\partial x\,\partial y} - \frac{\partial^2 F_2}{\partial x\,\partial z} + \frac{\partial^2 F_1}{\partial y\,\partial z} - \frac{\partial^2 F_3}{\partial y\,\partial x} + \frac{\partial^2 F_2}{\partial z\,\partial x} - \frac{\partial^2 F_1}{\partial z\,\partial y} \equiv 0$$

The divergence of the gradient of a scalar function is the **Laplacian**:

$$\operatorname{div}\left(\operatorname{grad} f\right) \equiv \vec{\boldsymbol{\nabla}}\bullet\vec{\boldsymbol{\nabla}}f \equiv \nabla^2 f \equiv \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

for all twice-differentiable scalar functions f.

In orthogonal non-Cartesian coordinate systems, the expressions for the gradient operator are not as simple.

5.02 Differentiation in Orthogonal Curvilinear Coordinate Systems

For any orthogonal curvilinear coordinate system (u_1, u_2, u_3) in $\mathbb{R}^3$,
the unit tangent vectors along the curvilinear axes are $\hat{\mathbf{e}}_i = \hat{\mathbf{T}}_i = \frac{1}{h_i}\frac{\partial \vec{\mathbf{r}}}{\partial u_i}$,

where the scale factors $\boxed{h_i = \left|\frac{\partial \vec{\mathbf{r}}}{\partial u_i}\right|}$.

The displacement vector $\vec{\mathbf{r}}$ can then be written as $\vec{\mathbf{r}} = u_1\hat{\mathbf{e}}_1 + u_2\hat{\mathbf{e}}_2 + u_3\hat{\mathbf{e}}_3$,
where the unit vectors $\hat{\mathbf{e}}_i$ form an **orthonormal basis** for $\mathbb{R}^3$.

$$\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = \delta_{ij} = \begin{cases} 0 & (i \neq j) \\ 1 & (i = j) \end{cases}$$

The differential displacement vector **dr** is (by the Chain Rule)

$$\mathbf{d}\vec{\mathbf{r}} = \frac{\partial \vec{\mathbf{r}}}{\partial u_1} du_1 + \frac{\partial \vec{\mathbf{r}}}{\partial u_2} du_2 + \frac{\partial \vec{\mathbf{r}}}{\partial u_3} du_3 = h_1\, du_1\, \hat{\mathbf{e}}_1 + h_2\, du_2\, \hat{\mathbf{e}}_2 + h_3\, du_3\, \hat{\mathbf{e}}_3$$

and the differential arc length ds is given by

$$ds^2 = \mathbf{d}\vec{\mathbf{r}} \bullet \mathbf{d}\vec{\mathbf{r}} = (h_1\, du_1)^2 + (h_2\, du_2)^2 + (h_3\, du_3)^2$$

The element of volume dV is

$$dV = h_1 h_2 h_3\, du_1 du_2 du_3 = \underbrace{\left|\frac{\partial(x, y, z)}{\partial(u_1, u_2, u_3)}\right|}_{\text{Jacobian}} du_1 du_2 du_3$$

$$= \begin{vmatrix} \frac{\partial x}{\partial u_1} & \frac{\partial y}{\partial u_1} & \frac{\partial z}{\partial u_1} \\ \frac{\partial x}{\partial u_2} & \frac{\partial y}{\partial u_2} & \frac{\partial z}{\partial u_2} \\ \frac{\partial x}{\partial u_3} & \frac{\partial y}{\partial u_3} & \frac{\partial z}{\partial u_3} \end{vmatrix} du_1 du_2 du_3$$

Example 5.02.1: Find the scale factor h_θ for the spherical polar coordinate system $(x, y, z) = (r \sin\theta \cos\phi,\ r \sin\theta \sin\phi,\ r\cos\theta)$:

$$\frac{\partial \vec{\mathbf{r}}}{\partial \theta} = \begin{bmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} r\cos\theta\cos\phi & r\cos\theta\sin\phi & -r\sin\theta \end{bmatrix}^{\mathrm{T}}$$

$$\Rightarrow\ h_\theta = \left|\frac{\partial \vec{\mathbf{r}}}{\partial \theta}\right| = \sqrt{r^2\cos^2\theta\cos^2\phi + r^2\cos^2\theta\sin^2\phi + r^2\sin^2\theta}$$

$$= \sqrt{r^2\cos^2\theta\,\cancel{(\cos^2\phi + \sin^2\phi)} + r^2\sin^2\theta} = \sqrt{r^2(\cos^2\theta + \sin^2\theta)} = r$$

5.03 Summary Table for the Gradient Operator

Gradient operator $$\vec{\nabla} = \frac{\hat{\mathbf{e}}_1}{h_1}\frac{\partial}{\partial u_1} + \frac{\hat{\mathbf{e}}_2}{h_2}\frac{\partial}{\partial u_2} + \frac{\hat{\mathbf{e}}_3}{h_3}\frac{\partial}{\partial u_3}$$

Gradient $$\vec{\nabla} V = \frac{\hat{\mathbf{e}}_1}{h_1}\frac{\partial V}{\partial u_1} + \frac{\hat{\mathbf{e}}_2}{h_2}\frac{\partial V}{\partial u_2} + \frac{\hat{\mathbf{e}}_3}{h_3}\frac{\partial V}{\partial u_3}$$

Divergence $$\vec{\nabla} \bullet \vec{\mathbf{F}} = \frac{1}{h_1 h_2 h_3}\left(\frac{\partial(h_2 h_3 F_1)}{\partial u_1} + \frac{\partial(h_3 h_1 F_2)}{\partial u_2} + \frac{\partial(h_1 h_2 F_3)}{\partial u_3}\right)$$

Curl $$\vec{\nabla} \times \vec{\mathbf{F}} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1 \hat{\mathbf{e}}_1 & \frac{\partial}{\partial u_1} & h_1 F_1 \\ h_2 \hat{\mathbf{e}}_2 & \frac{\partial}{\partial u_2} & h_2 F_2 \\ h_3 \hat{\mathbf{e}}_3 & \frac{\partial}{\partial u_3} & h_3 F_3 \end{vmatrix}$$

Laplacian $$\nabla^2 V = \frac{1}{h_1 h_2 h_3}\left(\frac{\partial}{\partial u_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial V}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3 h_1}{h_2}\frac{\partial V}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial V}{\partial u_3}\right)\right)$$

Scale factors:

Cartesian: $h_x = h_y = h_z = 1$.

Cylindrical polar: $h_\rho = h_z = 1$, $h_\phi = \rho$.

Spherical polar: $h_r = 1$, $h_\theta = r$, $h_\phi = r\sin\theta$.

Example 5.03.1: The Laplacian of V in spherical polars is

$$\nabla^2 V = \frac{1}{r^2 \sin\theta}\left(\frac{\partial}{\partial r}\left(r^2 \sin\theta \frac{\partial V}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{\partial}{\partial \phi}\left(\frac{1}{\sin\theta}\frac{\partial V}{\partial \phi}\right)\right)$$

or $$\nabla^2 V = \frac{\partial^2 V}{\partial r^2} + \frac{2}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\left(\frac{\partial^2 V}{\partial \theta^2} + \cot\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}$$

<u>Example 5.03.2</u>

A potential function $V(\vec{\mathbf{r}})$ is spherically symmetric, (that is, its value depends only on the distance r from the origin), due solely to a point source at the origin. There are no other sources or sinks anywhere in $\mathbb{R}^3$. Deduce the functional form of $V(\vec{\mathbf{r}})$.

$V(\vec{\mathbf{r}})$ is spherically symmetric $\Rightarrow$ $V(r, \theta, \phi) = f(r)$

In any regions not containing any sources of the vector field, the divergence of the vector field $\vec{\mathbf{F}} = \vec{\nabla} V$ (and therefore the Laplacian of the associated potential function V) must be zero. Therefore, for all $r \neq 0$, $\operatorname{div} \vec{\mathbf{F}} = \vec{\nabla} \bullet \vec{\nabla} V = \nabla^2 V = 0$

But

$$\nabla^2 V = \frac{1}{r^2 \sin\theta}\left(\frac{\partial}{\partial r}\left(r^2 \sin\theta \frac{\partial V}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{\partial}{\partial \phi}\left(\frac{1}{\sin\theta} \frac{\partial V}{\partial \phi}\right)\right)$$

$$\Rightarrow \quad \nabla^2 V = \frac{1}{r^2 \cancel{\sin\theta}}\left(\frac{d}{dr}\left(r^2 \cancel{\sin\theta} \frac{dV}{dr}\right) + 0 + 0\right) = 0$$

$$\Rightarrow \quad \frac{d}{dr}\left(r^2 \frac{dV}{dr}\right) = 0 \quad \Rightarrow \quad r^2 \frac{dV}{dr} = B \quad \Rightarrow \quad \frac{dV}{dr} = B r^{-2}$$

$\Rightarrow \quad V = \dfrac{B r^{-1}}{-1} + A$, where A, B are arbitrary constants of integration.

Therefore the potential function must be of the form

$$\boxed{V(r, \theta, \phi) = A - \frac{B}{r}}$$

This is the standard form of the potential function associated with a force that obeys the inverse square law $F \propto \dfrac{1}{r^2}$.

5.04 Derivatives of Basis Vectors

Cartesian: $\frac{d}{dt}\hat{\mathbf{i}} = \frac{d}{dt}\hat{\mathbf{j}} = \frac{d}{dt}\hat{\mathbf{k}} = \vec{\mathbf{0}}$

$$\vec{\mathbf{r}} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$$
$$\Rightarrow \quad \vec{\mathbf{v}} = \dot{x}\hat{\mathbf{i}} + \dot{y}\hat{\mathbf{j}} + \dot{z}\hat{\mathbf{k}}$$

Cylindrical Polar Coordinates:

$x = \rho\cos\phi, \quad y = \rho\sin\phi, \quad z = z$

$$\frac{d}{dt}\hat{\boldsymbol{\rho}} = \frac{d\phi}{dt}\hat{\boldsymbol{\phi}}$$
$$\frac{d}{dt}\hat{\boldsymbol{\phi}} = -\frac{d\phi}{dt}\hat{\boldsymbol{\rho}}$$
$$\frac{d}{dt}\hat{\mathbf{k}} = \vec{\mathbf{0}}$$

$$\mathbf{r} = \rho\,\hat{\boldsymbol{\rho}} + z\,\hat{\mathbf{k}}$$
$$\Rightarrow \quad \vec{\mathbf{v}} = \dot{\rho}\,\hat{\boldsymbol{\rho}} + \rho\dot{\phi}\hat{\boldsymbol{\phi}} + \dot{z}\,\hat{\mathbf{k}}$$

[radial and transverse components of $\vec{\mathbf{v}}$]

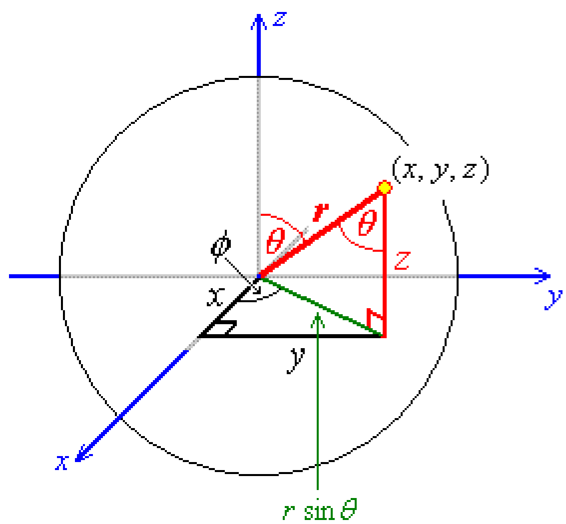
Spherical Polar Coordinates.

The "declination" angle θ is the angle between the positive z axis and the radius vector $\vec{\mathbf{r}}$. $0 \le \theta \le \pi$.

The "azimuth" angle ϕ is the angle on the x-y plane, measured anticlockwise from the positive x axis, of the shadow of the radius vector. $0 \le \phi < 2\pi$.

$$z = r\cos\theta\,.$$

The shadow of the radius vector on the x-y plane has length $r\sin\theta$.



It then follows that

$$x = r\sin\theta\cos\phi \quad \text{and} \quad y = r\sin\theta\sin\phi\,.$$

$$\frac{d}{dt}\hat{\mathbf{r}} = \frac{d\theta}{dt}\hat{\boldsymbol{\theta}} + \frac{d\phi}{dt}\sin\theta\,\hat{\boldsymbol{\phi}}$$
$$\frac{d}{dt}\hat{\boldsymbol{\theta}} = -\frac{d\theta}{dt}\hat{\mathbf{r}} + \frac{d\phi}{dt}\cos\theta\,\hat{\boldsymbol{\phi}}$$
$$\frac{d}{dt}\hat{\boldsymbol{\phi}} = -\frac{d\phi}{dt}\left(\sin\theta\,\hat{\mathbf{r}} + \cos\theta\,\hat{\boldsymbol{\theta}}\right)$$

$$\vec{\mathbf{r}} = r\,\hat{\mathbf{r}}$$
$$\Rightarrow \quad \vec{\mathbf{v}} = \dot{r}\,\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}} + r\dot{\phi}\sin\theta\,\hat{\boldsymbol{\phi}}$$

Example 5.04.1

Find the velocity and acceleration in cylindrical polar coordinates for a particle travelling along the helix $x = 3 \cos 2t$, $y = 3 \sin 2t$, $z = t$.

Cylindrical polar coordinates: $x = \rho \cos\phi$, $y = \rho \sin\phi$, $z = z$

$$\Rightarrow \quad \rho^2 = x^2 + y^2 , \quad \tan\phi = \frac{y}{x}$$

$$\rho^2 = 9\cos^2 2t + 9\sin^2 2t = 9 \quad \Rightarrow \quad \rho = 3 \quad \Rightarrow \quad \dot{\rho} = 0$$

$$\tan\phi = \frac{3\sin 2t}{3\cos 2t} = \tan 2t \quad \Rightarrow \quad \phi = 2t \quad \Rightarrow \quad \dot{\phi} = 2$$

$$z = t \quad \Rightarrow \quad \dot{z} = 1$$

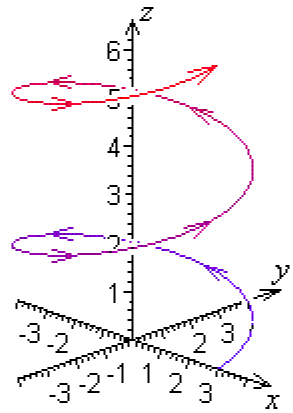
$$\Rightarrow \quad \vec{\mathbf{r}} = 3\hat{\boldsymbol{\rho}} + z\hat{\mathbf{k}}$$

$$\Rightarrow \quad \vec{\mathbf{v}} = \frac{d\vec{\mathbf{r}}}{dt} = \dot{\rho}\hat{\boldsymbol{\rho}} + \rho\dot{\phi}\hat{\boldsymbol{\phi}} + \dot{z}\hat{\mathbf{k}} = 0\hat{\boldsymbol{\rho}} + 3\times 2\hat{\boldsymbol{\phi}} + 1\hat{\mathbf{k}} = \underline{\underline{6\hat{\boldsymbol{\phi}} + \hat{\mathbf{k}}}}$$

[The velocity has no radial component – the helix remains the same distance from the z axis at all times.]

$$\vec{\mathbf{a}} = \frac{d\vec{\mathbf{v}}}{dt} = 6\dot{\hat{\boldsymbol{\phi}}} + \dot{\hat{\mathbf{k}}} = -6\dot{\phi}\hat{\boldsymbol{\rho}} + \vec{\mathbf{0}} = \underline{\underline{-12\hat{\boldsymbol{\rho}}}}$$

[The acceleration vector points directly at the z axis at all times.]



Other examples are in the problem sets.

END OF CHAPTER 5