Example 8.04.4

Find the complete solution to

$$\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x\,\partial y} + \frac{\partial^2 u}{\partial y^2} = 0,$$

$u = 0$ on $x = 0$,
$u = x^2$ on $y = 1$.

$A = 1, \quad B = 2, \quad C = 1 \quad \Rightarrow \quad D = 4 - 4\times 1 = 0$

Therefore the PDE is **parabolic** everywhere.

$$\lambda = \frac{-2 \pm \sqrt{0}}{2} = -1 \text{ or } -1$$

The complementary function (and general solution) is

$$u(x,y) = f(y-x) + h(x,y)\,g(y-x)$$

where $h(x, y)$ is any convenient non-trivial linear function of (x, y) except a multiple of $(y - x)$. Choosing, arbitrarily, $h(x, y) = x$,

$$u(x,y) = f(y-x) + x\,g(y-x)$$

Imposing the boundary conditions:

$u(0, y) = 0 \quad \Rightarrow \quad f(y) + 0 = 0$
Therefore the function f is identically zero, for any argument including $(y - x)$.

We now have $u(x, y) = x\,g(y - x)$.

$u(x, 1) = x^2 \quad \Rightarrow \quad x\,g(1 - x) = x^2 \quad \Rightarrow \quad g(1 - x) = x \quad \Rightarrow \quad g(x) = 1 - x$

Therefore
$u(x, y) = x\,g(y - x) = x\,(1 - (y - x))$

The complete solution is

$$\boxed{u(x,y) = x(x-y+1)}$$

Example 8.04.4 - Alternative Treatment of the Complementary Function

Find the complete solution to

$$\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x\,\partial y} + \frac{\partial^2 u}{\partial y^2} = 0,$$

$u = 0$ on $x = 0$,
$u = x^2$ on $y = 1$.

$A = 1,\quad B = 2,\quad C = 1 \quad\Rightarrow\quad D = 4 - 4\times 1 = 0$

$$\lambda = \frac{-2 \pm \sqrt{0}}{2} = -1 \text{ or } -1$$

The complementary function (and general solution) is

$$u(x,y) = f(y-x) + h(x,y)\,g(y-x)$$

where $h(x, y)$ is any convenient non-trivial linear function of (x, y) except a multiple of $(y - x)$. The most general choice possible is $h(x,y) = ax + by$, with the restriction $a \neq \lambda b$.

$$u(x,y) = f(y-x) + (ax+by)\cdot g(y-x)$$

$$\Rightarrow\quad u(0,y) = f(y) + by\cdot g(y) = 0 \quad\Rightarrow\quad f(y) = -by\cdot g(y) \qquad \textbf{(A)}$$

[note: choosing $b = 0 \;\rightarrow\; f(y) = 0$, as happened on the previous page] and

$$u(x,1) = f(1-x) + (ax+b)\cdot g(1-x) = x^2$$

Using **(A)**,

$$u(x,1) = -b(1-x)\cdot g(1-x) + (ax+b)\cdot g(1-x) = x^2$$

$$\Rightarrow\quad (\cancel{-b} + bx + ax \cancel{+b})\cdot g(1-x) = x^2 \quad\Rightarrow\quad (a+b)\,x\cdot g(1-x) = x^2$$

$$\Rightarrow\quad g(1-x) = \frac{x}{a+b}$$ (note that the restriction on $h(x, y)$ ensures that $a+b\neq 0$).

$$\Rightarrow\quad g(y) = \frac{1-y}{a+b} \qquad \textbf{(B)}$$

(A) $\Rightarrow\quad f(y) = -by\cdot\dfrac{1-y}{a+b}$

The complete solution becomes

$$u(x,y) = -b(y-x)\cdot\frac{1-(y-x)}{a+b} + (ax+by)\frac{1-(y-x)}{a+b}$$

$$= (\cancel{-by} + bx + ax \cancel{+by})\frac{1-(y-x)}{a+b} = \cancel{(a+b)}\,x\,\frac{x-y+1}{\cancel{a+b}} \quad\Rightarrow$$

$$\boxed{u(x,y) = x(x-y+1)}$$

Example 8.04.4 Extension (continued)

It is easy to confirm that this solution is correct:

$$u(x, y) = x(x - y + 1) \quad \Rightarrow$$
$$u(0, y) = 0(0 - y + 1) = 0 \quad \forall y \qquad ✓$$
and
$$u(x,1) = x(x - 1 + 1) = x^2 \quad \forall x \qquad ✓$$
and
$$u(x, y) = x^2 - xy + x \quad \Rightarrow \quad \frac{\partial u}{\partial x} = 2x - y \quad \text{and} \quad \frac{\partial u}{\partial y} = -x$$
$$\Rightarrow \quad \frac{\partial^2 u}{\partial x^2} = 2, \quad \frac{\partial^2 u}{\partial x\, \partial y} = -1 \quad \text{and} \quad \frac{\partial^2 u}{\partial y^2} = 0$$
$$\Rightarrow \quad \frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x\, \partial y} + \frac{\partial^2 u}{\partial y^2} = 2 - 2 + 0 = 0 \quad \forall (x, y) \qquad ✓$$

Two-dimensional Laplace Equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$A = C = 1\,, \quad B = 0 \quad \Rightarrow \quad D = 0 - 4 < 0$

This PDE is **elliptic** everywhere.

$$\lambda = \frac{0 \pm \sqrt{-4}}{2} = \pm j$$

The general solution is

$$u(x,y) = f(y - jx) + g(y + jx)$$

where f and g are any twice-differentiable functions.

A function $f(x, y)$ is **harmonic** if and only if $\nabla^2 f = 0$ everywhere inside a domain Ω.

<u>Example 8.04.5</u>

Is $u = e^x \sin y$ harmonic on $\mathbb{R}^2$?

$$\frac{\partial u}{\partial x} = e^x \sin y \quad \text{and} \quad \frac{\partial u}{\partial y} = e^x \cos y$$

$$\Rightarrow \quad \frac{\partial^2 u}{\partial x^2} = e^x \sin y \quad \text{and} \quad \frac{\partial^2 u}{\partial y^2} = -e^x \sin y$$

$$\Rightarrow \quad \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^x \sin y - e^x \sin y = 0 \qquad \forall (x, y)$$

Therefore **yes**, $u = e^x \sin y$ is harmonic on $\mathbb{R}^2$.

<u>Example 8.04.6</u>

Find the complete solution $u(x, y)$ to the partial differential equation $\nabla^2 u = 0$, given the additional information

$$u(0, y) = y^3 \quad \text{and} \quad \left.\frac{\partial u}{\partial x}\right|_{x=0} = 0$$

The PDE is

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

(which means that the solution $u(x, y)$ is an harmonic function).

$\Rightarrow \quad A = C = 1, \quad B = 0 \qquad \Rightarrow \quad D = B^2 - 4AC = -4 < 0$

The PDE is **elliptic** everywhere.

A.E.: $\lambda^2 + 1 = 0 \qquad \Rightarrow \quad \lambda = \pm j$

C.F.: $u_C(x, y) = f(y - jx) + g(y + jx)$

The PDE is homogeneous $\Rightarrow$

P.S.: $u_P(x, y) = 0$

G.S.: $u(x, y) = f(y - jx) + g(y + jx)$

$\Rightarrow \quad u_x(x, y) = -j f'(y - jx) + j g'(y + jx)$

Using the additional information,

$u(0, y) = f(y) + g(y) = y^3 \qquad \Rightarrow \quad g(y) = y^3 - f(y) \quad \Rightarrow \quad g'(y) = 3y^2 - f'(y)$

and

$u_x(0, y) = 0 = -j f'(y) + j g'(y) = j\left(-f'(y) + 3y^2 - f'(y)\right)$

$\Rightarrow \quad 2f'(y) = 3y^2 \qquad \Rightarrow \quad 2f(y) = y^3 \quad \Rightarrow \quad f(y) = \tfrac{1}{2}y^3$

$\Rightarrow \quad g(y) = y^3 - \tfrac{1}{2}y^3 = \tfrac{1}{2}y^3$

Therefore the complete solution is

$u(x, y) = \tfrac{1}{2}(y - jx)^3 + \tfrac{1}{2}(y + jx)^3$

$= \tfrac{1}{2}\left(y^3 - \cancel{3(jx)y^2} + 3(jx)^2 y - \cancel{(jx)^3} + y^3 + \cancel{3(jx)y^2} + 3(jx)^2 y + \cancel{(jx)^3}\right)$

$= y^3 + 3j^2 x^2 y \quad \Rightarrow$

$$\boxed{u(x, y) = y^3 - 3x^2 y}$$

Note that the solution is completely real, even though the eigenvalues are not real.

8.05 The Maximum-Minimum Principle

Let Ω be some finite domain on which a function $u(x, y)$ and its second derivatives are defined. Let $\overline{\Omega}$ be the union of the domain with its boundary.
Let m and M be the minimum and maximum values respectively of u on the boundary of the domain.

If $\nabla^2 u \geq 0$ in Ω, then u is **subharmonic** and

$$u(\vec{\mathbf{r}}) < M \quad \text{or} \quad u(\vec{\mathbf{r}}) \equiv M \qquad \forall \vec{\mathbf{r}} \text{ in } \Omega$$

If $\nabla^2 u \leq 0$ in Ω, then u is **superharmonic** and

$$u(\vec{\mathbf{r}}) > m \quad \text{or} \quad u(\vec{\mathbf{r}}) \equiv m \qquad \forall \vec{\mathbf{r}} \text{ in } \Omega$$

If $\nabla^2 u = 0$ in Ω, then u is **harmonic** (both subharmonic and superharmonic) and u is either constant on $\overline{\Omega}$ or $m < u < M$ everywhere on Ω.

Example 8.05.1

$\nabla^2 u = 0$ in $\Omega : x^2 + y^2 < 1$ and $u(x, y) = 1$ on $C : x^2 + y^2 = 1$.
Find $u(x, y)$ on Ω.

---

u is harmonic on $\Omega \quad \Rightarrow \quad \min_C (u) \leq \begin{pmatrix} u(x,y) \\ \text{on } \Omega \end{pmatrix} \leq \max_C (u)$

But $\min_C(u) = \max_C(u) = 1$

Therefore **$u(x, y) = 1$ everywhere in Ω**.

Example 8.05.2

$\nabla^2 u = 0$ in the square domain $\Omega : -2 < x < +2,\ -2 < y < +2$.
On the boundary C, on the left and right edges ($x = \pm 2$), $u(x, y) = 4 - y^2$,
while on the top and bottom edges ($y = \pm 2$), $u(x, y) = x^2 - 4$.

Find bounds on the value of $u(x, y)$ inside the domain Ω.

For $-2 \le y \le +2,\ 0 \le 4 - y^2 \le 4$.
For $-2 \le x \le +2,\ -4 \le x^2 - 4 \le 0$.
Therefore, on the boundary C of the domain Ω, $-4 \le u(x, y) \le +4$ so that
$m = -4$ and $M = +4$.

$u(x, y)$ is harmonic (because $\nabla^2 u = 0$).

Therefore, everywhere in Ω,

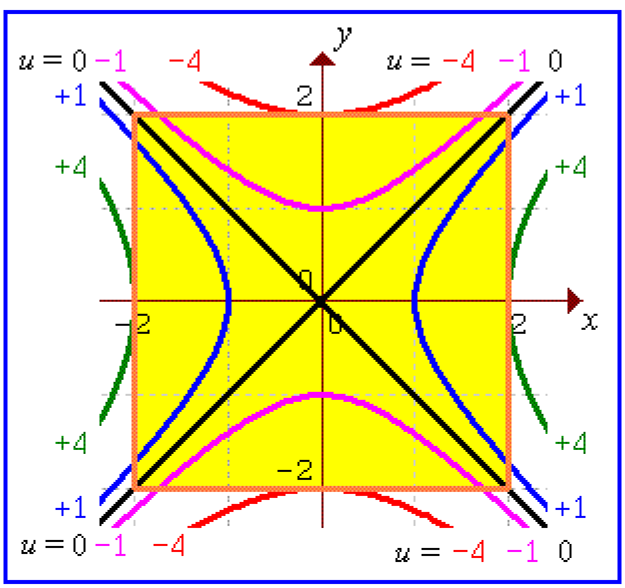
$$-4 < u(x, y) < +4$$

Note:
$u(x, y) = x^2 - y^2$ is consistent with the boundary condition and

$$\frac{\partial u}{\partial x} = 2x - 0\,, \quad \frac{\partial u}{\partial y} = 0 - 2y \quad \Rightarrow \quad \frac{\partial^2 u}{\partial x^2} = 2\,, \quad \frac{\partial^2 u}{\partial y^2} = -2 = -\frac{\partial^2 u}{\partial x^2}$$

$$\Rightarrow \quad \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Contours of constant values of u are hyperbolas.
A contour map illustrates that $-4 < u(x, y) < +4$ within the domain is indeed true.



8.06 The Heat Equation

For a material of constant density ρ, constant specific heat μ and constant thermal conductivity K, the partial differential equation governing the temperature u at any location (x, y, z) and any time t is

$$\frac{\partial u}{\partial t} = k \nabla^2 u \quad , \quad \text{where} \quad k = \frac{K}{\mu\rho}$$

Example 8.06.1

Heat is conducted along a thin homogeneous bar extending from $x = 0$ to $x = L$. There is no heat loss from the sides of the bar. The two ends of the bar are maintained at temperatures T_1 (at $x = 0$) and T_2 (at $x = L$). The initial temperature throughout the bar at the cross-section x is $f(x)$.

Find the temperature at any point in the bar at any subsequent time.

The partial differential equation governing the temperature $u(x, t)$ in the bar is

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$$

together with the boundary conditions

$u(0,t) = T_1$ and $u(L,t) = T_2$

and the initial condition

$u(x, 0) = f(x)$

[Note that if an end of the bar is insulated, instead of being maintained at a constant temperature, then the boundary condition changes to $\frac{\partial u}{\partial x}(0,t) = 0$ or $\frac{\partial u}{\partial x}(L,t) = 0$.]

Attempt a solution by the method of separation of variables.

$u(x, t) = X(x)\, T(t)$

$$\Rightarrow \quad X T' = k X'' T \qquad \Rightarrow \quad \frac{T'}{T} = k \frac{X''}{X} = c$$

Again, when a function of t only equals a function of x only, both functions must equal the same absolute constant. Unfortunately, the two boundary conditions cannot both be satisfied unless $T_1 = T_2 = 0$. Therefore we need to treat this more general case as a perturbation of the simpler ($T_1 = T_2 = 0$) case.

Example 8.06.1 (continued)

Let $u(x, t) = v(x, t) + g(x)$
Substitute this into the PDE:

$$\frac{\partial}{\partial t}\big(v(x,t)+g(x)\big) = k\,\frac{\partial^2}{\partial x^2}\big(v(x,t)+g(x)\big) \;\Rightarrow\; \frac{\partial v}{\partial t} = k\left(\frac{\partial^2 v}{\partial x^2} + g''(x)\right)$$

This is the standard heat PDE for v if we choose g such that $g''(x) = 0$.
$g(x)$ must therefore be a linear function of x.

We want the perturbation function $g(x)$ to be such that

$$u(0,t) = T_1 \quad \text{and} \quad u(L,t) = T_2$$

and

$$v(0, t) = v(L, t) = 0$$

Therefore $g(x)$ must be the linear function for which $g(0) = T_1$ and $g(L) = T_2$.
It follows that

$$g(x) = \left(\frac{T_2 - T_1}{L}\right)x + T_1$$

and we now have the simpler problem

$$\frac{\partial v}{\partial t} = k\,\frac{\partial^2 v}{\partial x^2}$$

together with the boundary conditions

$$v(0, t) = v(L, t) = 0$$

and the initial condition

$$v(x, 0) = f(x) - g(x)$$

Now try separation of variables on $v(x, t)$:
$v(x, t) = X(x)\,T(t)$

$$\Rightarrow \quad X\,T' = k\,X''T \qquad \Rightarrow \quad \frac{1}{k}\frac{T'}{T} = \frac{X''}{X} = c$$

But $v(0, t) = v(L, t) = 0 \;\Rightarrow\; X(0) = X(L) = 0$

This requires c to be a negative constant, say $-\lambda^2$.
The solution is very similar to that for the wave equation on a finite string with fixed ends (section 8.02). The eigenvalues are $\lambda = \dfrac{n\pi}{L}$ and the corresponding eigenfunctions are any non-zero constant multiples of

$$X_n(x) = \sin\left(\frac{n\pi x}{L}\right)$$

Example 8.06.1 (continued)

The ODE for $T(t)$ becomes

$$T' + \left(\frac{n\pi}{L}\right)^2 kT = 0$$

whose general solution is

$$T_n(t) = c_n e^{-n^2\pi^2 kt/L^2}$$

Therefore

$$v_n(x,t) = X_n(x)T_n(t) = c_n \sin\left(\frac{n\pi x}{L}\right)\exp\left(-\frac{n^2\pi^2 kt}{L^2}\right)$$

If the initial temperature distribution $f(x) - g(x)$ is a simple multiple of $\sin\left(\frac{n\pi x}{L}\right)$ for some integer n, then the solution for v is just $v(x,t) = c_n \sin\left(\frac{n\pi x}{L}\right)\exp\left(-\frac{n^2\pi^2 kt}{L^2}\right)$.

Otherwise, we must attempt a superposition of solutions.

$$v(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right)\exp\left(-\frac{n^2\pi^2 kt}{L^2}\right)$$

such that $v(x,0) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) = f(x) - g(x)$.

The Fourier sine series coefficients are $c_n = \frac{2}{L}\int_0^L \left(f(z) - g(z)\right)\sin\left(\frac{n\pi z}{L}\right)dz$

so that the complete solution for $v(x, t)$ is

$$v(x,t) = \frac{2}{L}\sum_{n=1}^{\infty}\left(\int_0^L \left(f(z) - \frac{T_2 - T_1}{L}z - T_1\right)\sin\left(\frac{n\pi z}{L}\right)dz\right)\sin\left(\frac{n\pi x}{L}\right)\exp\left(-\frac{n^2\pi^2 kt}{L^2}\right)$$

and the complete solution for $u(x, t)$ is

$$u(x,t) = v(x,t) + \left(\frac{T_2 - T_1}{L}\right)x + T_1$$

Note how this solution can be partitioned into a transient part $v(x, t)$ (which decays to zero as t increases) and a steady-state part $g(x)$ which is the limiting value that the temperature distribution approaches.

Example 8.06.1 (continued)

As a specific example, let $k = 9$, $T_1 = 100$, $T_2 = 200$, $L = 2$ and
$f(x) = 145x^2 - 240x + 100$, (for which $f(0) = 100$, $f(2) = 200$ and $f(x) > 0 \;\forall x$).

Then $g(x) = \dfrac{200-100}{2}x + 100 = 50x + 100$

The Fourier sine series coefficients are

$$c_n = \int_0^2 \left(\left(145z^2 - 240z + 100\right) - \left(50z + 100\right) \right) \sin\left(\frac{n\pi z}{2}\right) dz$$

$$\Rightarrow \quad c_n = 145 \int_0^2 \left(z^2 - 2z\right) \sin\left(\frac{n\pi z}{2}\right) dz$$

After an integration by parts (details omitted here),

$$\Rightarrow \quad c_n = 145 \left[\left(\left(-z^2 + 2z\right) \frac{2}{n\pi} + \frac{16}{(n\pi)^3} \right) \cos\left(\frac{n\pi z}{2}\right) + \frac{8(z-1)}{(n\pi)^2} \sin\left(\frac{n\pi z}{2}\right) \right]_{z=0}^{z=2}$$

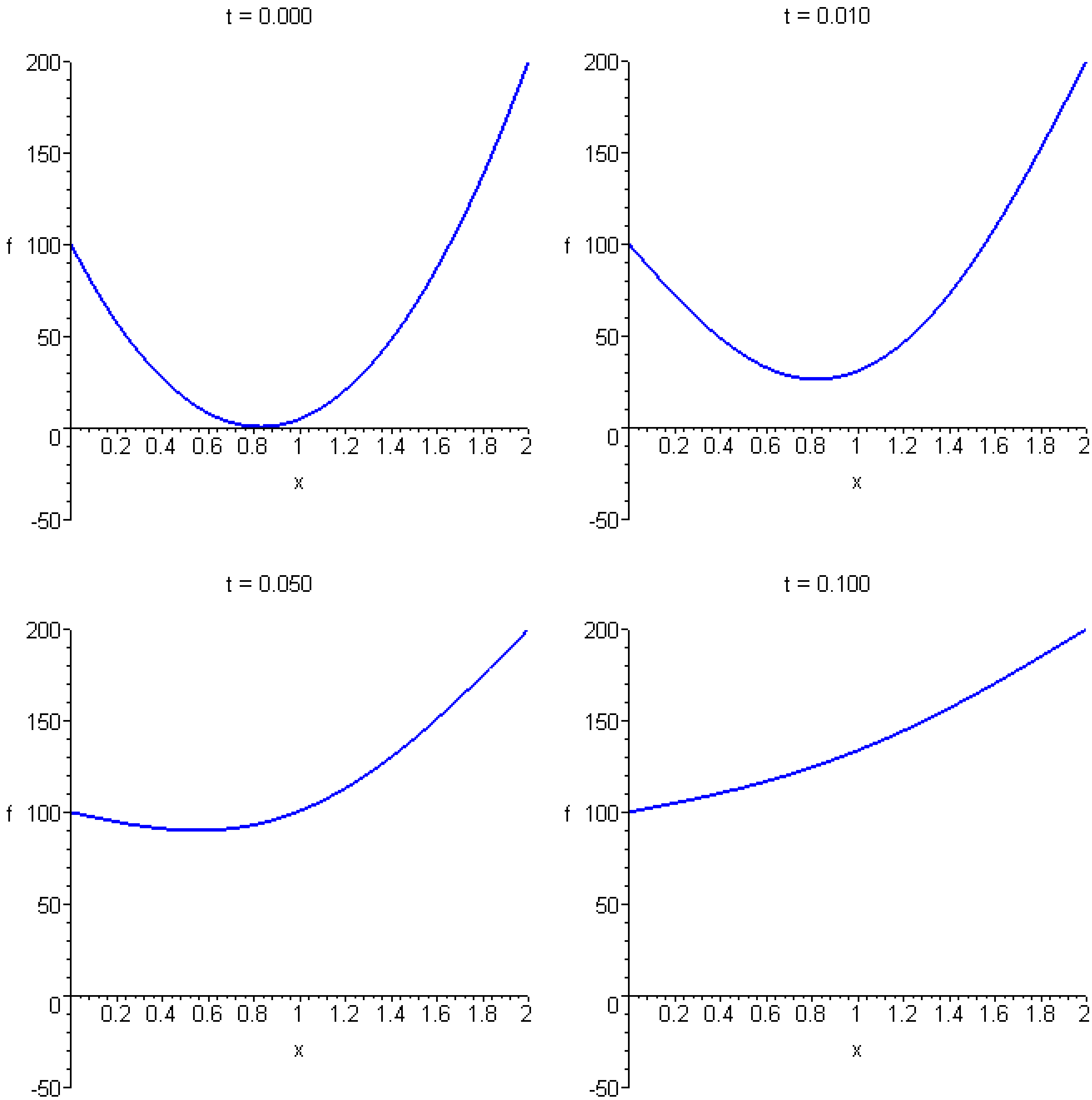
$$\Rightarrow \quad c_n = \frac{2320}{(n\pi)^3} \left((-1)^n - 1 \right)$$

The complete solution is

$$u(x,t) = 50x + 100 - \frac{2320}{\pi^3} \sum_{n=1}^{\infty} \left(\frac{1 - (-1)^n}{n^3} \right) \sin\left(\frac{n\pi x}{2}\right) \exp\left(-\frac{9n^2\pi^2 t}{4} \right)$$

Some snapshots of the temperature distribution (from the tenth partial sum) from the Maple file at "`www.engr.mun.ca/~ggeorge/9420/demos/ex8061.mws`" are shown on the next page.

Example 8.06.1 (continued)



The steady state distribution is nearly attained in much less than a second!

END OF CHAPTER 8
END OF ENGI. 9420!