ENGI 9420 Engineering Analysis
Assignment 1 Solutions

2012 Fall
[First order ODEs, Sections 1.01-1.05]
Note*:** In this assignment, do ***not use Laplace transform methods at all.

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1. For the initial value problem

$$\frac{dy}{dx} + 2y = 2, \qquad y(0) = 4$$

(a) Classify the ODE (as one or more of separable, exact, linear, or Bernoulli). [2]

This ODE is not exact: $(2y-2)\,dx + dy = 0$

$$P = 2y-2 \;\Rightarrow\; \frac{\partial P}{\partial y} = 2, \qquad Q = 1 \;\Rightarrow\; \frac{\partial Q}{\partial x} = 0 \neq \frac{\partial P}{\partial y}$$

However, [from part (b) below], the ODE is

both separable and linear

[*All* homogeneous first order linear ODEs are special cases of Bernoulli ODEs with $n = 0$.]

(b) Obtain the complete solution by two different methods. [10]

<u>Method of Separation of Variables</u>:

$$\frac{dy}{dx} = 2-2y \;\Rightarrow\; dy = 2(1-y)\,dx \;\Rightarrow\; \frac{1}{2}\int \frac{dy}{1-y} = \int dx$$

$$\Rightarrow\; -\frac{1}{2}\ln|1-y| = x + c_1 \;\Rightarrow\; \ln|1-y| = -2x + c_2$$

$$\Rightarrow\; 1-y = \pm e^{-2x+c_2} = c_3 e^{-2x} \;\Rightarrow\; y(x) = 1 - c_3 e^{-2x}$$

where c_1, c_2 and c_3 are all arbitrary constants.

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1 (b) (continued)

Substitute the initial condition into this general solution:
$y(0)=4 \quad \Rightarrow \quad 4 = 1-c_3 \quad \Rightarrow \quad c_3 = -3$
Therefore the complete solution is

$$\boxed{y(x) = 1 + 3e^{-2x}}$$

AND

Linear Method:

$$\frac{dy}{dx} + \underset{\substack{\uparrow \\ P}}{2}\,y = \underset{\substack{\uparrow \\ R}}{2}$$

$$h = \int P\,dx = \int 2\,dx = 2x \qquad \Rightarrow \quad e^h = e^{2x}$$

$$\Rightarrow \quad \int e^h R\,dx = \int 2e^{2x}\,dx = e^{2x}$$

Therefore the general solution of the ODE is

$$y = e^{-h}\left(\int e^h R\,dx + C\right) = e^{-2x}\left(e^{2x} + C\right) = 1 + C e^{-2x}$$

Substitute the initial condition into this general solution:
$y(0)=4 \quad \Rightarrow \quad 4 = 1+C \quad \Rightarrow \quad C = 3$
Therefore the complete solution is

$$\boxed{y(x) = 1 + 3e^{-2x}}$$

(c) Verify that your solution does satisfy the initial value problem. [4]

$$y(x) = 1 + 3e^{-2x} \qquad \Rightarrow \quad y'(x) = 0 - 6e^{-2x} \qquad ✓$$

$$\Rightarrow \quad y' + 2y = -6e^{-2x} + \left(2 + 6e^{-2x}\right) = 2$$

and $y(0) = 1 + 3 = 4$ ✓

2. For the initial value problem

$$\frac{dy}{dx} + 2y = 2y^3, \qquad y(0) = 4$$

(a) Classify the ODE (as one or more of separable, exact, linear, or Bernoulli). [2]

Rewrite the ODE in standard form in order to test for exactness:

$$\frac{dy}{dx} + 2y - 2y^3 = 0 \quad \Rightarrow \quad \underbrace{2y\left(1-y^2\right)}_{P}dx + \underbrace{1}_{Q}\,dy = 0$$

$$\Rightarrow \quad \frac{\partial P}{\partial y} = 2\left(1-3y^2\right), \qquad \frac{\partial Q}{\partial x} = 0 \neq \frac{\partial P}{\partial y}$$

Therefore the ODE is not exact.
In the original form of the ODE, we can quickly identify
$P = R = 2,\ \ n = 3 \ \Rightarrow$ the ODE is Bernoulli, but not linear.
$n > 0 \ \Rightarrow$ the singular solution $y \equiv 0$ also exists.
P and R are both constants $\Rightarrow$ the ODE is also separable.
The ODE is therefore

both separable and Bernoulli

(b) Obtain the complete solution. [5]

Either one of the following methods may be used:

Method of Separation of Variables:

$$\frac{dy}{dx} = 2y^3 - 2y \quad \Rightarrow \quad \int \frac{dy}{2y\left(y^2-1\right)} = \int dx$$

unless $y \equiv 0$ or $y \equiv -1$ or $y \equiv +1$
(all of which may be singular solutions to the ODE).
Using the cover-up rule for partial fractions,

$$\frac{1}{y(y-1)(y+1)} = \frac{\left(\dfrac{1}{\cancel{0} \times -1 \times 1}\right)}{y} + \frac{\left(\dfrac{1}{1 \times \cancel{0} \times 2}\right)}{y-1} + \frac{\left(\dfrac{1}{-1 \times -2 \times \cancel{0}}\right)}{y+1}$$

$$\Rightarrow \quad \int \frac{dy}{2y(y-1)(y+1)} = -\frac{1}{2}\int \frac{dy}{y} + \frac{1}{4}\int \frac{dy}{y-1} + \frac{1}{4}\int \frac{dy}{y+1} = \int dx$$

2 (b) (continued)

$$\Rightarrow \quad -\frac{1}{2}\ln|y| + \frac{1}{4}\ln|y-1| + \frac{1}{4}\ln|y+1| = x + c_1$$

$$\Rightarrow \quad \frac{1}{4}\ln\left|\frac{y^2-1}{y^2}\right| = x + c_1 \qquad \Rightarrow \quad \frac{y^2-1}{y^2} = \pm e^{4x+c_2} = c_3 e^{4x}$$

$$\Rightarrow \quad y^2 - 1 = c_3 y^2 e^{4x} \qquad \Rightarrow \quad y^2\left(1 - c_3 e^{4x}\right) = 1$$

$$\Rightarrow \quad y^2 = \frac{1}{1 - c_3 e^{4x}} \qquad \Rightarrow \quad y(x) = \pm\sqrt{\frac{1}{1 - c_3 e^{4x}}}$$

But the initial condition is positive, so the positive square root is required.
Also note that $y(0) = 4$ is incompatible with any of the three singular solutions.
Substitute the initial condition into the general solution:

$$y(0) = 4 \quad \Rightarrow \quad 4 = +\sqrt{\frac{1}{1 - c_3}} \quad \Rightarrow \quad 1 - c_3 = \frac{1}{16} \quad \Rightarrow c_3 = \frac{15}{16}$$

Therefore the complete solution to the ODE is

$$\boxed{y(x) = +\sqrt{\frac{1}{1 - \frac{15}{16} e^{4x}}}}$$

OR

Bernoulli Method:

Let $w = \dfrac{y^{1-n}}{1-n} = \dfrac{y^{1-3}}{1-3} = -\dfrac{1}{2y^2}$,

then the Bernoulli ODE for y transforms into the linear ODE for w:

$$\frac{dw}{dx} + (1-n)P\,w = R \qquad \Rightarrow \quad \frac{dw}{dx} - 4w = 2$$

$$h = \int -4\,dx \qquad \Rightarrow \quad e^h = e^{-4x}$$

$$\Rightarrow \quad \int e^h R\,dx = \int 2e^{-4x}\,dx = -\frac{e^{-4x}}{2}$$

$$\Rightarrow \quad -\frac{1}{2y^2} = w = e^{-h}\left(\int e^h R\,dx + C\right) = e^{4x}\left(C - \frac{e^{-4x}}{2}\right)$$

$$\Rightarrow \quad -\frac{1}{\not{2}y^2} = \frac{2Ce^{4x} - 1}{\not{2}} \qquad \Rightarrow \quad y^2 = \frac{1}{1 + Ae^{4x}}$$

But the initial condition is positive, so the positive square root is required.
Also note that $y(0) = 4$ is incompatible with the singular solution $y \equiv 0$.

2 (b) (continued)

Substitute the initial condition into the general solution:

$$y(0)=4 \quad \Rightarrow \quad 4 \;=\; +\sqrt{\frac{1}{1+A}} \quad \Rightarrow \quad 1+A \;=\; \frac{1}{16} \quad \Rightarrow A \;=\; -\frac{15}{16}$$

Therefore the complete solution to the ODE is

$$\boxed{y(x) \;=\; +\sqrt{\frac{1}{1-\frac{15}{16}e^{4x}}}}$$

[Additional note on singular solutions:
The constant solutions $y \equiv -1$ and $y \equiv +1$ arise from the general solution of the Bernoulli ODE upon setting the arbitrary constant $A = 0$ and are therefore not truly singular.
However there is ***no*** value of A in the general solution for which the singular solution $y \equiv 0$ is possible.]

(c) Verify that your solution does satisfy the initial value problem. [4]

$$y(x) \;=\; \left(1-\tfrac{15}{16}e^{4x}\right)^{-1/2} \quad \Rightarrow \quad y'(x) \;=\; -\frac{1}{2}\left(1-\tfrac{15}{16}e^{4x}\right)^{-3/2}\left(-\frac{15}{4}e^{4x}\right)$$

$$\Rightarrow \quad y' + 2y \;=\; \frac{15}{8}e^{4x}\left(1-\tfrac{15}{16}e^{4x}\right)^{-3/2} \;+\; 2\left(1-\tfrac{15}{16}e^{4x}\right)^{-1/2}$$

$$= \frac{2\left(\tfrac{15}{16}e^{4x} + \left(1-\tfrac{15}{16}e^{4x}\right)\right)}{\left(1-\tfrac{15}{16}e^{4x}\right)^{3/2}} \;=\; \frac{2}{\left(1-\tfrac{15}{16}e^{4x}\right)^{3/2}} \;=\; 2y^3 \quad ✓$$

$$y(0) \;=\; \sqrt{\frac{1}{1-\frac{15}{16}}} \;=\; \sqrt{\frac{1}{\frac{1}{16}}} \;=\; \sqrt{16} \;=\; 4 \quad ✓$$

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2 (d) Find the complete solution when the initial condition is replaced by $y(0) = 0$. [4]

There is no finite value of the arbitrary constant in the general solution in part (b) above which will yield a zero value of y for any finite value of x. However the initial condition here is consistent with the singular solution. Therefore the complete solution in this case is

$$\boxed{y \equiv 0}$$

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3. For the ordinary differential equation

$$y\,dx + (2x+3y)\,dy = 0$$

(a) Show that the ODE is not exact. [2]

$$P = y \quad \Rightarrow \quad \frac{\partial P}{\partial y} = 1$$

$$Q = 2x+3y \quad \Rightarrow \quad \frac{\partial Q}{\partial x} = 2 \neq \frac{\partial P}{\partial y}$$

Therefore the ODE is not exact.

(b) Find an integrating factor for this ODE. [4]

Try for an integrating factor as a function of x only:

$$\frac{\dfrac{\partial P}{\partial y} - \dfrac{\partial Q}{\partial x}}{Q} = \frac{1-2}{2x+3y} = \frac{-1}{2x+3y} \neq R(x)$$

Therefore the integrating factor cannot be a function of x only.

Try for an integrating factor as a function of y only:

$$\frac{\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}}{P} = \frac{2-1}{y} = \frac{1}{y} = R(y)$$

$$\Rightarrow \int R(y)\,dy = \int \frac{1}{y}\,dy = \ln y \qquad \Rightarrow \quad I(y) = e^{\ln y}$$

Therefore the integrating factor is

$$\boxed{I(y) = y}$$

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3 (c) Hence find the general solution (in implicit form). [6]

The exact form of the ODE is

$$y^2\,dx + \left(2xy + 3y^2\right)dy = 0$$

Seek a potential function $u(x, y)$ such that

$$\frac{\partial u}{\partial x} = y^2 \quad \text{and} \quad \frac{\partial u}{\partial y} = 2xy + 3y^2$$

$$\Rightarrow \quad u(x,y) = xy^2 + y^3 = c$$

This can be expressed in the explicit form

$$x = \frac{c - y^3}{y^2}$$

However, the following implicit form is an acceptable final answer:

$$\boxed{xy^2 + y^3 = A}$$

[Implicit differentiation of this solution quickly verifies that it does satisfy the original ODE:

$$\frac{d}{dx}\left(xy^2 + y^3\right) = 1y^2 + x\left(2y\frac{dy}{dx}\right) + 3y^2\frac{dy}{dx} = 0$$

$$\Rightarrow \quad y \equiv 0 \quad \text{or} \quad y\,dx + (2x + 3y)\,dy = 0 \,]$$

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4. Find the complete solution to the initial value problem [12]

$$\frac{dy}{dx} + 3y = 3y^{2/3}, \quad y(0) = 0$$

The ODE is separable and it is Bernoulli $\left(n = \frac{2}{3}\right)$.

Separation of variables:

$$\frac{dy}{dx} + 3y = 3y^{2/3} \quad \Rightarrow \quad \frac{dy}{dx} = 3y^{2/3} - 3y \quad \Rightarrow \quad \frac{dy}{3\left(y^{2/3} - y\right)} = dx \quad \text{or} \quad y \equiv 0$$

The singular solution $y \equiv 0$ is consistent with the initial condition.

Try to rearrange the left integrand into the form $\dfrac{f'(y)}{f(y)}$:

$$\frac{1}{3\left(y^{2/3} - y\right)} = \frac{1}{3y^{2/3}\left(1 - y^{1/3}\right)} = \frac{\frac{1}{3}y^{-2/3}}{1 - y^{1/3}} = -\frac{d}{dy}\left(\ln\left(1 - y^{1/3}\right)\right)$$

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4. (continued)

$$\int \frac{dy}{3\left(y^{2/3} - y\right)} = \int 1\,dx \quad \Rightarrow \quad -\ln\left(1 - y^{1/3}\right) = x + C$$

$$y(0) = 0 \quad \Rightarrow \quad -\ln(1-0) = 0 + C \quad \Rightarrow \quad C = 0$$

$$\Rightarrow \quad \ln\left(1 - y^{1/3}\right) = -x \quad \Rightarrow \quad 1 - y^{1/3} = e^{-x} \quad \Rightarrow \quad y^{1/3} = 1 - e^{-x}$$

Therefore the complete solution is

$$\boxed{y = \left(1 - e^{-x}\right)^3 \text{ or } y \equiv 0}$$

OR

Bernoulli ODE (with $n = \frac{2}{3}$):

The change of variables $w = \dfrac{y^{1-n}}{1-n} = 3y^{1/3}$ transforms the ODE into the linear form

$$\frac{dw}{dx} + 3 \times \frac{1}{3} w = 3 \quad \Rightarrow \quad \frac{dw}{dx} + 1w = 3$$

$$h = \int 1\,dx = x \qquad \Rightarrow \quad e^h = e^x \quad \text{(integrating factor)}$$

$$\Rightarrow \quad \int e^h R\,dx = \int e^x 3\,dx = 3e^x$$

$$\Rightarrow \quad 3y^{1/3} = w = e^{-h}\left(\int e^h R\,dx + C\right) = e^{-x}\left(3e^x + C\right)$$

$\Rightarrow \quad y^{1/3} = 1 + A\,e^{-x}$ and the solution $y \equiv 0$ cannot be obtained for any choice of A.

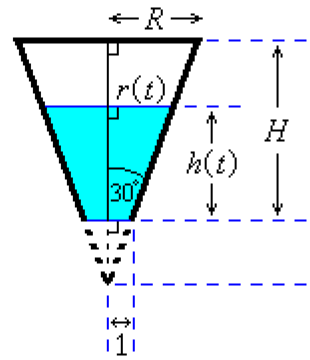
But $y(0) = 0 \quad \Rightarrow \quad 0 = 1 + A \quad \Rightarrow \quad A = -1$

$$\Rightarrow \quad y^{1/3} = 1 - e^{-x} \quad \Rightarrow \quad y = \left(1 - e^{-x}\right)^3$$

Therefore the complete solution is

$$\boxed{y = \left(1 - e^{-x}\right)^3 \text{ or } y \equiv 0}$$

5. A conical tank, of half-angle 30°, contains liquid as shown. The apex has been cut off to leave a circular hole of radius 1 centimetre, through which liquid drains out from the container. The "head" (or height) of liquid above the hole at any instant t is $h(t)$. The tank has a radius at the top of R and a height from the top to the hole of H. All distances are measured in centimetres.



The rate at which the volume $V(t)$ of liquid in the tank changes due to liquid draining at discharge speed $v(t)$ through a hole of area A is given by the differential equation

[20]

$$\frac{dV}{dt} = -kAv$$

where k is an experimentally determined constant, (dependent on viscosity and the geometry of the opening), between 0 and 1. For this question, assume $k = 0.7$.

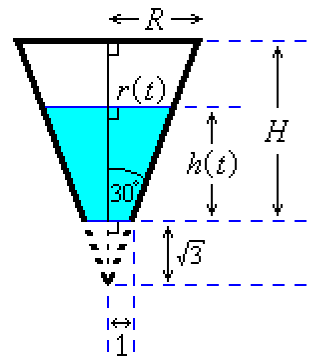
In addition, Toricelli's law (equating the gain of kinetic energy of every point in the water to the loss of gravitational potential energy of that point) leads to

$$v(t) = \sqrt{2g\,h(t)}$$

Find how long it takes (T) for a full tank to drain completely, as a function of the truncated height H of the cone.

Hence find the value of T to the nearest second when $H = 30$ cm.
Take $g = 981\,\text{cm}\,\text{s}^{-2}$.

5. (continued)



The distance from the hole to the apex of the cone is a, where

$$\frac{1}{a} = \tan 30° = \frac{1}{\sqrt{3}} \quad \Rightarrow \quad a = \sqrt{3}$$

The area of the circular hole, in units of cm^2, is

$$A = \pi(1)^2 = \pi$$

The differential equation for the volume, incorporating Toricelli's law, is

$$\frac{dV}{dt} = -kAv = -0.7\pi\sqrt{2g\,h(t)}$$

The volume of a cone of radius a and height b is $V = \frac{1}{3}\pi a^2 b$

The volume of liquid in the tank at any instant t is:

$$V(t) = \tfrac{1}{3}\pi r^2\left(h(t)+\sqrt{3}\right) - \tfrac{1}{3}\pi(1)^2\sqrt{3}$$

But, from the geometry of similar triangles,

$$\frac{h+\sqrt{3}}{r} = \frac{\sqrt{3}}{1} = \frac{H+\sqrt{3}}{R} \quad \Rightarrow \quad h+\sqrt{3} = r\sqrt{3} \text{ and } H+\sqrt{3} = R\sqrt{3}$$

$$\Rightarrow \quad V(t) = \frac{\pi\sqrt{3}}{3}\left(\left(r(t)\right)^3 - 1\right) \quad \text{and} \quad \frac{dV}{dt} = -0.7\pi\sqrt{2g\sqrt{3}(r-1)}$$

<u>Method 1</u> (using r as the independent variable):

By the chain rule,

$$V(t) = \frac{\pi\sqrt{3}}{3}\left(r^3-1\right) \quad \Rightarrow \quad \frac{dV}{dt} = \frac{dV}{dr}\cdot\frac{dr}{dt} = \pi r^2\sqrt{3}\,\frac{dr}{dt}$$

Equate the two expressions for dV/dt :

$$\frac{dV}{dt} = -0.7\pi\sqrt{2g\sqrt{3}(r-1)} = \pi r^2\sqrt{3}\,\frac{dr}{dt} \quad \Rightarrow \quad \frac{dr}{dt} = \frac{-0.7\sqrt{2g\sqrt{3}(r-1)}}{r^2\sqrt{3}}$$

$$\Rightarrow \quad \frac{r^2}{\sqrt{r-1}}\,dr = -0.7\sqrt{\frac{2g}{\sqrt{3}}}\,dt \quad \Rightarrow \quad -\int_R^1 \frac{r^2}{\sqrt{r-1}}\,dr = +0.7\sqrt{\frac{2g}{\sqrt{3}}}\int_0^T 1\,dt$$

Note that $\int_0^T 1\,dt = \left[t\right]_0^T = T - 0 = T$.

5. (continued)

Use the change of variables $u = r - 1 \quad \Rightarrow \quad du = dr$

$$\Rightarrow \quad -\int_R^1 \frac{r^2}{\sqrt{r-1}}\,dr \;=\; -\int_{R-1}^0 \frac{(u+1)^2}{\sqrt{u}}\,du \;=\; +\int_0^{R-1} \frac{u^2+2u+1}{\sqrt{u}}\,du$$

$$= \int_0^{R-1} \left(u^{3/2} + 2u^{1/2} + u^{-1/2}\right) du \;=\; \left[\frac{2u^{5/2}}{5} + \frac{4u^{3/2}}{3} + 2u^{1/2}\right]_0^{R-1}$$

$$= \left[\frac{6u^{5/2} + 20u^{3/2} + 30u^{1/2}}{15}\right]_0^{R-1} \;=\; \frac{6(R-1)^{5/2} + 20(R-1)^{3/2} + 30(R-1)^{1/2}}{15} - 0$$

$$\Rightarrow \quad T \;=\; \frac{10}{7\times 15}\sqrt{\frac{\sqrt{3}}{2g}}\left(6(R-1)^{5/2} + 20(R-1)^{3/2} + 30(R-1)^{1/2}\right)$$

$$\Rightarrow \quad T \;=\; \frac{1}{21}\sqrt{\frac{2(R-1)\sqrt{3}}{g}}\left(6(R-1)^2 + 20(R-1) + 30\right)$$

$$H \;=\; (R-1)\sqrt{3} \qquad \Rightarrow \quad T \;=\; \frac{1}{21}\sqrt{\frac{2H}{g}}\left(2H^2 + 20\frac{H}{\sqrt{3}} + 30\right) \quad \text{or}$$

$$\boxed{T \;=\; \frac{1}{21}\sqrt{\frac{2H}{3g}}\left(2\sqrt{3}H^2 + 20H + 30\sqrt{3}\right)}$$

OR

<u>Method 2</u> (using h as the independent variable):

$$V(t) \;=\; \frac{\pi\sqrt{3}}{3}\left(\left(r(t)\right)^3 - 1\right) \;=\; \frac{\pi\sqrt{3}}{3}\left(\left(1 + \frac{h(t)}{\sqrt{3}}\right)^3 - 1\right)$$

$$\Rightarrow \quad \frac{dV}{dt} \;=\; \pi\left(1 + \frac{h}{\sqrt{3}}\right)^2 \frac{dh}{dt} \;=\; -0.7\pi\sqrt{2gh}$$

$$\Rightarrow \quad \frac{1}{\sqrt{h}}\left(1 + \frac{h}{\sqrt{3}}\right)^2 \frac{dh}{dt} \;=\; -0.7\sqrt{2g} \quad \Rightarrow \quad \frac{1 + \frac{2}{\sqrt{3}}h + \frac{1}{3}h^2}{\sqrt{h}}\,dh \;=\; -0.7\sqrt{2g}\;dt$$

$$\Rightarrow \quad -\int_H^0 \left(h^{-1/2} + \tfrac{2}{\sqrt{3}}h^{1/2} + \tfrac{1}{3}h^{3/2}\right) dh \;=\; +\frac{7}{10}\sqrt{2g}\int_0^T 1\,dt$$

5. (continued)

$$\Rightarrow \quad +\left[2h^{1/2} + \frac{4h^{3/2}}{3\sqrt{3}} + \frac{2h^{5/2}}{15}\right]_0^H = +\frac{7}{10}\sqrt{2g}\,\Big[t\Big]_0^T$$

$$\Rightarrow \quad T = \frac{10}{7}\sqrt{\frac{H}{2g}} \cdot \frac{30\sqrt{3} + 20H + 2\sqrt{3}H^2}{15\sqrt{3}} \quad \text{or}$$

$$\boxed{T = \frac{1}{21}\sqrt{\frac{2H}{3g}}\left(2\sqrt{3}H^2 + 20H + 30\sqrt{3}\right)}$$

Replacing g by 981 cm s^{-2} and H by 30 cm, we find $T = 25.6308...$ s, or, to the nearest second,

$$\boxed{T = 26\text{ s}}$$

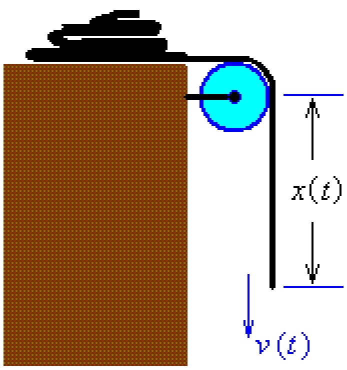
An Excel file is available here. It displays values of T for various choices of H.

6. A five metre long chain with a constant line density of ρ kg/m is supported in a pile on a platform several metres above the floor of a warehouse. It is wound around a frictionless pulley at the edge of the platform, with one metre of chain already hanging down at time $t = 0$, when the chain is released from rest. Let $x(t)$ represent the length of that part of the chain hanging down from the pulley at time t and let $v(t)$ be the speed of that part of the chain at time t.

(a) Show that the ordinary differential equation governing the speed of the chain is [8]

$$\frac{dv}{dx} + \frac{1}{x}v = \frac{g}{v}, \qquad (1 \le x \le 5)$$

where $g \approx 9.81\,\text{m}\,\text{s}^{-2}$ is the acceleration due to gravity and where all frictional forces are ignored.



Newton's second law:

$$\frac{d}{dt}(mv) = W$$

But $m = \rho x \ne$ constant $\qquad W = mg = \rho x g$

$$\Rightarrow \frac{d}{dt}(\rho x)\cdot v + \rho x \frac{dv}{dt} = \rho x g$$

But $\dfrac{dx}{dt} = v$ and $\dfrac{dv}{dt} = \dfrac{dv}{dx}\cdot\dfrac{dx}{dt} = v\dfrac{dv}{dx}$

$$\Rightarrow v^2 + xv\frac{dv}{dx} = xg$$

Also, the chain starts at $x = 1$ and the trailing end leaves the pulley when $x = 5$. Therefore the governing ODE is

$$\frac{dv}{dx} + \frac{1}{x}v = \frac{g}{v}, \qquad (1 \le x \le 5)$$

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6 (b) Determine the speed with which the trailing end of the chain leaves the pulley. [17]

The ODE is a Bernoulli ODE, with $P = x - 1$, $R = g$ and $n = -1$.

Using the substitution $w = \dfrac{v^{1-(-1)}}{1-(-1)} = \dfrac{v^2}{2}$ the ODE transforms into the linear ODE

$$\frac{dw}{dx} + \frac{2}{x}w = g$$

$$h = \int \frac{2}{x}\,dx = 2\ln x = \ln\left(x^2\right) \quad \Rightarrow \quad e^h = x^2$$

$$\int e^h R\,dx = \int x^2 g\,dx = \frac{g x^3}{3}$$

$$\Rightarrow \quad \frac{v^2}{2} = w = e^{-h}\left(\int e^h R\,dx + C\right) = \frac{1}{x^2}\left(\frac{g x^3}{3} + C\right) = \frac{g x}{3} + \frac{C}{x^2}$$

But $v = 0$ when $x = 1$ $\quad \Rightarrow \quad 0 = \dfrac{g}{3} + C \quad \Rightarrow \quad C = -\dfrac{g}{3}$

The complete solution to the ODE, in implicit form, is

$$v^2 = \frac{2g}{3}\left(x - \frac{1}{x^2}\right)$$

$$x = 5 \quad \Rightarrow \quad v = \sqrt{\frac{2g}{3}\left(5 - \frac{1}{25}\right)} = \sqrt{\frac{2g}{3}\left(\frac{124}{25}\right)} = \frac{2}{5}\sqrt{\frac{62g}{3}}$$

Therefore, to 3 s.f., the speed with which the trailing end of the chain leaves the pulley is

$$\boxed{v = 5.70\ \text{m}\,\text{s}^{-1}}$$

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