ENGI 9420 Engineering Analysis
Assignment 6 Solutions

2012 Fall
[Calculus of variations, Fourier series; Chapters 6 and 7]

Reminder of the principal concepts of the **calculus of variations**:

The path $y = f(x)$ between the points $(a, f(a))$ and $(b, f(b))$ (where both the x and y coordinates of both points are absolute constants) that minimizes (or maximizes) the value of the integral

$$I = \int_a^b F\left(x, y(x), y'(x)\right) dx$$

must satisfy the Euler equation for extremals, $\dfrac{d}{dx}\left(\dfrac{\partial F}{\partial y'}\right) - \dfrac{\partial F}{\partial y} = 0$

or, equivalently,

$$y'' \frac{\partial^2 F}{\partial y'^2} + y' \frac{\partial^2 F}{\partial y\, \partial y'} + \left(\frac{\partial^2 F}{\partial x\, \partial y'} - \frac{\partial F}{\partial y}\right) = 0$$

Special cases of this equation are:

If F is not explicitly dependent on y then $\dfrac{\partial F}{\partial y'} = c$

If F is not explicitly dependent on x then $y' \dfrac{\partial F}{\partial y'} - F = c$

If F is explicitly dependent on neither x nor y then

$$y = Ax + B \quad \left(\text{provided } \frac{\partial^2 F}{\partial y'^2} \not\equiv 0\right)$$

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1 (a) Find extremals $y(x)$ for $I = \displaystyle\int_{x_0}^{x_1} \frac{3 + (y')^2}{x^4}\, dx$ [7]

$$F = \frac{3 + (y')^2}{x^4} \quad \Rightarrow \quad \frac{\partial F}{\partial y} = 0$$

The Euler equation for extremals then simplifies to

1. (continued)

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \quad \Rightarrow \quad \frac{d}{dx}\left(\frac{0+2y'}{x^4}\right) = 0 \quad \Rightarrow \quad \frac{2y'}{x^4} = c_1$$

$$\Rightarrow \quad y' = \frac{c_1}{2}x^4 \quad \Rightarrow \quad y = \frac{c_1}{10}x^5 + c_2$$

Redefining the two arbitrary constants of integration, the family of extremals is

$$\boxed{y = Ax^5 + B}$$

(b) Find the equation of the extremal that passes through the points (0, 3) and (1, 7). [5]

The extremal must pass through (0, 3) $\Rightarrow \quad 3 = 0 + B \quad \Rightarrow \quad B = 3$

The extremal must pass through (1, 7) $\Rightarrow \quad 7 = A + 3 \quad \Rightarrow \quad A = 4$

Therefore the unique extremal is

$$\boxed{y = 4x^5 + 3}$$

(c) Show that the extremal in part (b) minimizes the integral I. [8]

Let Γ_0 be the path $y = 4x^5 + 3$.

$\Gamma : y = 4x^5 + 3 + g(x)$ is any other continuous path from (0, 3) to (1, 7)

$\Rightarrow \quad g(0) = g(1) = 0$ and $g'(x) \not\equiv 0$

Comparing the values of I along the paths Γ and Γ_0:

$$I(\Gamma_0) = \int_0^1 \frac{3 + \left(20x^4 + 0\right)^2}{x^4}\,dx \quad \text{and} \quad I(\Gamma) = \int_0^1 \frac{3 + \left(20x^4 + 0 + g'(x)\right)^2}{x^4}\,dx$$

$$= \int_0^1 \frac{3 + \left(20x^4\right)^2}{x^4}\,dx + 2\int_0^1 \frac{20x^4 \cdot g'(x)}{x^4}\,dx + \int_0^1 \frac{\left(g'(x)\right)^2}{x^4}\,dx$$

$$\Rightarrow \quad I(\Gamma) = I(\Gamma_0) + 40\cancel{\left[g(x)\right]_0^1} + \int_0^1 [\text{positive}]\,dx > I(\Gamma_0) \quad \forall g(x)$$

Therefore $y = 4x^5 + 3$ minimizes the integral I.

2. Find the path between the points (1, 0) and (2, 1) along which a chain of minimum mass can be laid, when the density at every point on the chain is inversely proportional to the distance of that point from the *y* axis.
Classify (or describe) the geometric shape of this path.

Some key steps to the solution are provided here:

(a) Let ρ represent the line density (weight per unit length) of the chain. [5]
Then $\rho = \frac{k}{x}$, where k is a constant of proportionality.

Show that the total mass of the chain is $m = k \int_1^2 \frac{\sqrt{1+(y')^2}}{x} dx$.

This is the integral to be minimized.

In any sufficiently small section of the chain, the element of mass is related to the line density and the element of arc length by

$$\Delta m \approx \rho \, \Delta s \approx \frac{k}{x}\sqrt{(\Delta x)^2 + (\Delta y)^2} = \frac{k}{x}\sqrt{1+\left(\frac{\Delta y}{\Delta x}\right)^2}\,\Delta x$$

Integrating along the length of the chain between (1, 0) and (2, 1), the total mass of the chain is

$$m = k\int_1^2 \frac{\sqrt{1+(y')^2}}{x} dx$$

(b) Show that the solution to the Euler-Lagrange equation for this integral is the ordinary differential equation [5]

$$\frac{y'}{x\sqrt{1+(y')^2}} = c_1$$

where c_1 is an arbitrary constant of integration.

$$F(x, y, y') = \frac{\sqrt{1+(y')^2}}{x} \quad \Rightarrow \quad F \text{ is independent of } y.$$

The Euler-Lagrange equation $\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0$ simplifies to $\frac{\partial F}{\partial y'} = c_1$

$$\Rightarrow \quad \frac{\partial}{\partial y'}\left(\frac{\sqrt{1+(y')^2}}{x}\right) = \frac{y'}{x\sqrt{1+(y')^2}} = c_1$$

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2 (c) Show that the general solution to this ODE is [5]

$$y = \mp \frac{\sqrt{1 - c_1^2 x^2}}{c_1} + c_2$$

where c_2 is another arbitrary constant of integration.

$$\frac{y'}{x\sqrt{1+(y')^2}} = c_1 \quad \Rightarrow \quad y' = c_1 x\sqrt{1+(y')^2} \quad \Rightarrow \quad (y')^2 = c_1^2 x^2 \left(1+(y')^2\right)$$

$$\Rightarrow \quad \left(1 - c_1^2 x^2\right)(y')^2 = c_1^2 x^2 \quad \Rightarrow \quad (y')^2 = \frac{c_1^2 x^2}{1 - c_1^2 x^2}$$

$$\Rightarrow \quad y' = \pm \frac{c_1 x}{\sqrt{1 - c_1^2 x^2}}$$

Using the standard antiderivative $\displaystyle\int \frac{k\, f'(x)}{\sqrt{f(x)}}\, dx = 2k\sqrt{f(x)} + C$

with $f(x) = 1 - c_1^2 x^2 \quad \Rightarrow \quad f'(x) = -2c_1^2 x$, so that $2k = \mp 1$, the solution follows:

$$y = \mp \frac{\sqrt{1 - c_1^2 x^2}}{c_1} + c_2$$

(d) Use the fact that this path must pass through both of the points (1, 0) and (2, 1) to determine the values of the two arbitrary constants. This determines the path along which the chain must lie if it is to have an extremal value of mass among all chains that connect those two points. [5]

Path passes through (1, 0) $\quad \Rightarrow \quad 0 = \mp \dfrac{\sqrt{1 - c_1^2}}{c_1} + c_2 \quad \Rightarrow \quad c_2 = \pm \dfrac{\sqrt{1 - c_1^2}}{c_1}$

Path passes through (2, 1) $\quad \Rightarrow \quad 1 = \mp \dfrac{\sqrt{1 - 4c_1^2}}{c_1} + c_2 = \pm \left(\dfrac{\sqrt{1 - c_1^2}}{c_1} - \dfrac{\sqrt{1 - 4c_1^2}}{c_1} \right)$

$$\Rightarrow \quad c_1 = \pm\left(\sqrt{1 - c_1^2} - \sqrt{1 - 4c_1^2}\right) \quad \Rightarrow \quad c_1^2 = \left(\sqrt{1 - c_1^2} - \sqrt{1 - 4c_1^2}\right)^2$$

$$\Rightarrow \quad c_1^2 = 1 - c_1^2 + 1 - 4c_1^2 - 2\sqrt{1 - c_1^2}\sqrt{1 - 4c_1^2}$$

$$\Rightarrow \quad 6c_1^2 - 2 = -2\sqrt{1 - c_1^2}\sqrt{1 - 4c_1^2} \quad \Rightarrow \quad 3c_1^2 - 1 = -\sqrt{1 - c_1^2}\sqrt{1 - 4c_1^2}$$

$$\Rightarrow \quad 9c_1^4 - 6c_1^2 + 1 = \left(1 - c_1^2\right)\left(1 - 4c_1^2\right) = 1 - 5c_1^2 + 4c_1^4$$

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2 (d) (continued)

$$\Rightarrow \quad 5c_1^{\,4} - c_1^{\,2} = 0 \qquad \Rightarrow \quad c_1^{\,2}\left(5c_1^{\,2} - 1\right) = 0 \quad \Rightarrow \quad c_1^{\,2} = 0 \text{ or } \tfrac{1}{5}$$

$c_1 = 0$ corresponds to $y' \equiv 0 \quad \Rightarrow \quad y = c$
which is inconsistent with two different values of y at the two ends of the path.
Therefore $c_1^{\,2} = \tfrac{1}{5}$ only

$$\Rightarrow \quad c_2 = \pm\frac{\sqrt{1-\tfrac{1}{5}}}{\tfrac{1}{5}} = \pm\sqrt{5-1} = \pm 2$$

The only choice of signs for the pair of arbitrary constants that is consistent with $(x, y) = (1, 0)$ and $(x, y) = (2, 1)$ leads to the complete solution

$$\boxed{y = 2 - \sqrt{5 - x^2}}$$

(e) Explain why this extremal path must provide the *minimum* value of mass (and not a maximum). [5]

For the integral whose value is the mass m of the chain, $y = 2 - \sqrt{5 - x^2}$ is the only extremal. If any one other path results in a greater mass, then the extremal must be an absolute minimum.

Obviously, a very long chain between the two values of x will have a greater mass, especially if part of it lies near the y axis, where the density is infinite. Therefore the extremal path provides the minimum value for the mass of the chain.

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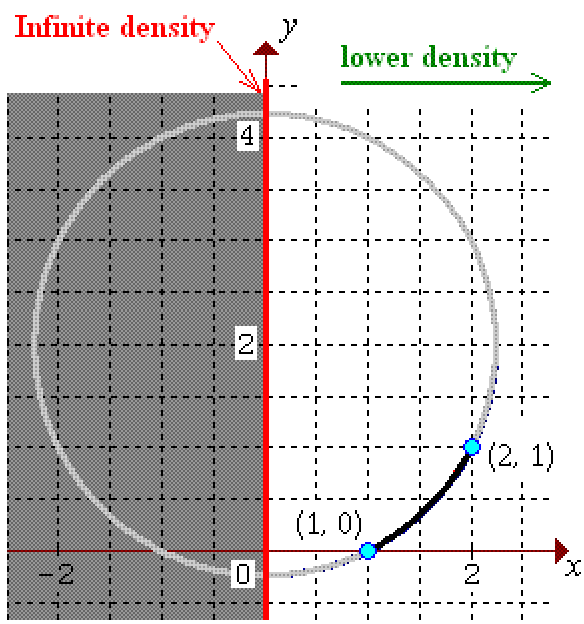
2 (f) Inspect the functional form of your solution $y(x)$ and deduce the geometric shape of this minimal-mass path. [5]

$$y = 2 - \sqrt{5 - x^2} \quad \Rightarrow \quad y - 2 = -\sqrt{5 - x^2}$$

$$\Rightarrow \quad (y-2)^2 = 5 - x^2 \quad \Rightarrow \quad x^2 + (y-2)^2 = 5$$

which is the equation of a circle. The negative sign on the square root term in the functional form for y tells us that the minimum-mass path must be an

arc on the lower half of the circle, centre (0, 2), radius $\sqrt{5}$.



3. Find the Fourier series expansion of the function [15]

$$f(x) = x - x^3 \qquad (-1 \le x < 1)$$

$f(x) = x - x^3$ is odd $\quad \Rightarrow \quad a_n = 0 \quad \forall n$

$$b_n = \frac{1}{1}\int_{-1}^{1} \left(x - x^3\right) \sin\left(\frac{n\pi x}{1}\right) dx$$

Integration by parts:

$$= \left[-\left(\frac{(n\pi)^2 \left(x - x^3\right) + 6x}{(n\pi)^3} \right) \cos(n\pi x) \right.$$

$$\left. + \cancel{\left(\frac{(n\pi)^2 \left(1 - 3x^2\right) + 6}{(n\pi)^4} \right) \sin(n\pi x)} \right]_{-1}^{1}$$

$$= \frac{\cos(n\pi)}{(n\pi)^3} \left(\left(-\cancel{(n\pi)^2}(0) - 6 \right) + \left(\cancel{(n\pi)^2}(-1+1) - 6 \right) \right)$$

$$= \frac{-12}{(n\pi)^3} (-1)^n$$

	D		I
	$x - x^3$		$\sin(n\pi x)$
		$+$	
	$1 - 3x^2$		$-\frac{\cos(n\pi x)}{n\pi}$
		$-$	
	$-6x$		$-\frac{\sin(n\pi x)}{(n\pi)^2}$
		$+$	
	-6		$\frac{\cos(n\pi x)}{(n\pi)^3}$
		$-$	
	0		$\frac{\sin(n\pi x)}{(n\pi)^4}$

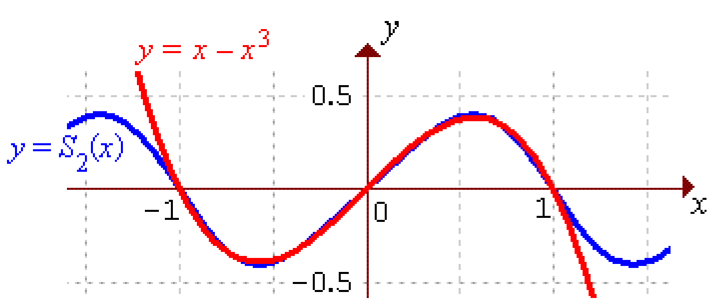
Therefore the Fourier series expansion of $x - x^3$ on [–1, +1) is

$$\boxed{f(x) = \frac{12}{\pi^3} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3} \sin(n\pi x)}$$

The first few terms are

$$f(x) = \frac{12}{\pi^3} \left(\sin(\pi x) - \frac{\sin(2\pi x)}{8} + \frac{\sin(3\pi x)}{27} - \frac{\sin(4\pi x)}{64} + \dots \right)$$

Note how good the agreement is on [–1, +1] between the function and the partial sum of only the first *two* terms of its Fourier series:



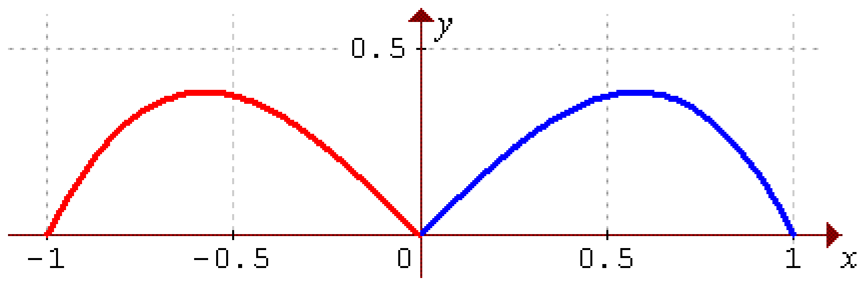
4. Find the Fourier cosine series expansion of the function [20]

$$f(x) = x - x^3 \qquad (0 < x < 1)$$

and

comment on the reasons for the differences in the approach to convergence of the series in questions (3) and (4) on the interval [0, 1].

In order to obtain a Fourier cosine series, we need an even extension.



Evaluating the Fourier cosine coefficients:

$$a_0 = \frac{2}{1}\int_0^1 \left(x - x^3\right) dx = \left[x^2 - \frac{x^4}{2}\right]_0^1$$

$$= \left(1 - \tfrac{1}{2}\right) - (0 - 0) = \tfrac{1}{2}$$

$$a_n = 2\int_0^1 \left(x - x^3\right)\cos\left(\frac{n\pi x}{1}\right) dx$$

Integration by parts:

$\underline{D}$	$\underline{I}$
$x - x^3$	$\cos(n\pi x)$
+	
$1 - 3x^2$	$\frac{\sin(n\pi x)}{n\pi}$
−	
$-6x$	$-\frac{\cos(n\pi x)}{(n\pi)^2}$
+	
-6	$-\frac{\sin(n\pi x)}{(n\pi)^3}$
−	
0	$\frac{\cos(n\pi x)}{(n\pi)^4}$

$$= 2\left[\cancel{\left(\frac{(n\pi)^2\left(x - x^3\right) + 6x}{(n\pi)^3}\right)\sin(n\pi x)}\right.$$

$$\left. + \left(\frac{(n\pi)^2\left(1 - 3x^2\right) + 6}{(n\pi)^4}\right)\cos(n\pi x)\right]_0^1$$

$$= \frac{2}{(n\pi)^4}\left(\left((n\pi)^2(1-3) + 6\right)\cos(n\pi) - \left((n\pi)^2(1-0) + 6\right)\right)$$

$$\Rightarrow \quad a_n = -2\left(\frac{1 + 2(-1)^n}{(n\pi)^2} + \frac{6\left(1 - (-1)^n\right)}{(n\pi)^4}\right) \quad \left(n \in \mathbb{N}\right)$$

Therefore the Fourier cosine series expansion of $x - x^3$ on [0, +1) is

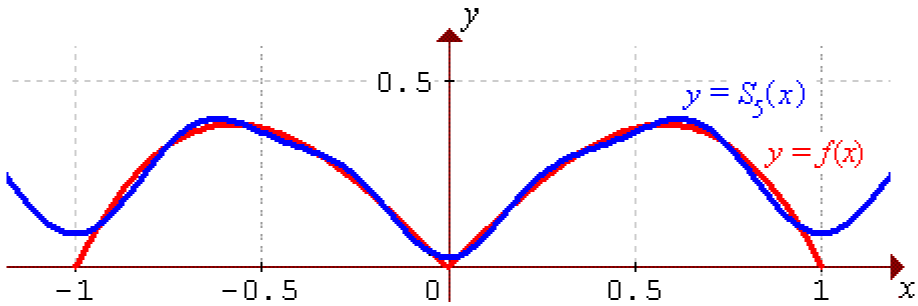
$$\boxed{f(x) = \frac{1}{4} - 2\sum_{n=1}^{\infty}\left(\frac{1 + 2(-1)^n}{(n\pi)^2} + \frac{6\left(1 - (-1)^n\right)}{(n\pi)^4}\right)\cos(n\pi x)}$$

4. (continued)

The first few terms are

$$f(x) = \frac{1}{4} + \left(\frac{2}{\pi^2} - \frac{24}{\pi^4}\right)\cos(\pi x) - \frac{3}{2\pi^2}\cos(2\pi x) + \left(\frac{2}{9\pi^2} - \frac{8}{27\pi^4}\right)\cos(3\pi x) + \ldots$$

The agreement, even for the fifth partial sum, is not that good near the endpoints $x = \pm 1$ and near the left endpoint $x = 0$ of the original interval.



The reason is that **the slope of the extended function is discontinuous at $x = 0$ and at $x = 1$**, whereas no such discontinuity occurs in question 3.

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5. Two continuous functions $f(x), g(x)$ that are integrable on an interval $[a, b]$ are said to be orthogonal on that interval if and only if their inner product is zero:

$$(f, g) = \int_a^b f(x)\, g(x)\, dx = 0$$

(a) Show that any integrable even function is orthogonal to any integrable odd function on any interval that is symmetric about zero. [7]

$$(f, g) = \int_{-L}^{L} f(x)\, g(x)\, dx$$

But the product of any even function with any odd function is an odd function. Integration of any integrable odd function over any finite interval that is symmetric about zero produces complete cancellation to a zero value. [Lecture notes, page 7.04].

$$\int_{-L}^{L} y(x)\, dx = \begin{cases} 0 & \text{if } y(x) \text{ is odd} \\ 2\int_0^L y(x)\, dx & \text{if } y(x) \text{ is even} \end{cases}$$

Therefore any integrable even function is orthogonal to any integrable odd function on any interval that is symmetric about zero

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5 (b) Find the relationship that the non-zero constants a and b must have if the odd function $f(x) = x$ is to be orthogonal to the odd function $g(x) = ax^3 + bx$ on the interval [–1, 1]. [8]

$$(f,g) = \int_{-1}^{1} (x)(ax^3 + bx)\,dx = \int_{-1}^{1} (ax^4 + bx^2)\,dx = 2\int_{0}^{1} (ax^4 + bx^2)\,dx$$

(because the integrand is even)

$$\Rightarrow \quad (f,g) = 2\left[\frac{ax^5}{5} + \frac{bx^3}{3}\right]_0^1 = 2\left(\frac{a}{5} + \frac{b}{3} - 0 - 0\right) = \frac{2}{15}(3a + 5b)$$

In order for these two functions to be orthogonal, we must have

$$\boxed{3a + 5b = 0}$$

[**Note**: The first two odd Legendre polynomials are $P_1(x) = x$ and $P_3(x) = \tfrac{1}{2}(5x^3 - 3x)$.
These two odd polynomials are orthogonal on [–1, 1].
All Legendre polynomials are such that $P_n(1) = 1 \quad (n = 0, 1, 2, \ldots)$.
In the same way, one can show that if $P_5(x) = ax^5 + bx^3 + cx$ must be orthogonal to both $P_3(x) = \tfrac{1}{2}(5x^3 - 3x)$ and $P_1(x) = x$ on [–1, 1],
then $10a + 9b = 0$ and $15a + 21b + 35c = 0$, leading to $a = -\tfrac{9}{10}b$ and $c = -\tfrac{3}{14}b$.
Imposing the additional requirement $P_5(1) = 1$ leads to $P_5(x) = \tfrac{1}{8}(63x^5 - 70x^3 + 15x)$.]

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