ENGI 9420 Engineering Analysis
Solutions to Additional Exercises

2012 Fall
[Partial differential equations; Chapter 8]

---

1 The function $u(x,y)$ satisfies $\dfrac{\partial^2 u}{\partial x^2} - 3\dfrac{\partial^2 u}{\partial x\,\partial y} + 2\dfrac{\partial^2 u}{\partial y^2} = 0$, subject to the boundary conditions $u(x,0) = \left.\dfrac{\partial}{\partial y}u(x,y)\right|_{y=0} = -1$. Classify the partial differential equation (hyperbolic, parabolic or elliptic) and find the complete solution $u(x,y)$.

Compare this PDE with the standard form for a d'Alembert solution

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x\,\partial y} + C\frac{\partial^2 u}{\partial y^2} = 0$$

$$A = 1, \quad B = -3, \quad C = 2 \qquad \Rightarrow \quad D = B^2 - 4AC = 9 - 8 = 1 > 0$$

This PDE is therefore

$$\boxed{\textbf{hyperbolic everywhere}}$$

Solving the characteristic (or auxiliary) equation:

$$\lambda = \frac{-B \pm \sqrt{D}}{2A} = \frac{+3 \pm \sqrt{1}}{2} = 1 \text{ or } 2$$

This PDE is homogeneous, so the complementary function is also the general solution:

$$u(x,y) = f(y+1x) + g(y+2x),$$

where f and g could be any twice-differentiable functions of their arguments.

Imposing the boundary conditions:

$$u(x,0) = f(x) + g(2x) = -1 \qquad \textbf{(A)}$$

$$u_y(x,y) = f'(y+x) + g'(y+2x)$$

$$u_y(x,0) = f'(x) + g'(2x) = -1 \qquad \textbf{(B)}$$

$$\frac{\partial}{\partial x}\textbf{(A)}: \quad f'(x) + 2g'(2x) = 0 \qquad \textbf{(C)}$$

$$\textbf{(C)} - \textbf{(B)}: \qquad g'(2x) = 1 \;\Rightarrow\; g'(x) = 1 \;\Rightarrow\; g(x) = x + k$$

$$\textbf{(A)} \;\Rightarrow \qquad f(x) = -1 - g(2x) = -1 - 2x - k$$

---

1. (continued)

Therefore the complete solution is

$u(x,y) = f(y+x) + g(y+2x) = \left(-1 - 2(y+x) - \cancel{k}\right) + \left((y+2x) + \cancel{k}\right)$

[Note that the arbitrary constant k always cancels out in this type of PDE solution.]

$\Rightarrow$

$$\boxed{u(x,y) = -y-1}$$

---

2. Classify the partial differential equation $4\frac{\partial^2 u}{\partial x^2} + 12\frac{\partial^2 u}{\partial x\,\partial y} + 9\frac{\partial^2 u}{\partial y^2} = 0$ and find its general solution.

$A=4, \quad B=12, \quad C=9 \qquad \Rightarrow \quad D = B^2 - 4AC = 144 - 144 = 0$

This PDE is therefore

parabolic everywhere

Solving the characteristic (or auxiliary) equation:

$\lambda = \frac{-B \pm \sqrt{D}}{2A} = \frac{-12 \pm \sqrt{0}}{8} = -\frac{3}{2} \text{ or } -\frac{3}{2}$

General solution:

$u(x,y) = f\left(y + \left(-\tfrac{3}{2}\right)x\right) + h(x,y)\,g\left(y + \left(-\tfrac{3}{2}\right)x\right),$

or equivalently

$u(x,y) = f(2y-3x) + h(x,y)\,g(2y-3x),$

where f and g could be any twice-differentiable functions of their arguments and h is any convenient non-trivial linear function of x and y except a multiple of $2y-3x$.

Arbitrarily choose $h = x$.

Then the general solution can be written as

$$\boxed{u(x,y) = f(2y-3x) + x\,g(2y-3x)}$$

[The functional forms of f and g remain arbitrary in the absence of any further information.]

3. A disturbance on a very long string causes a vertical displacement $y(x,t)$ at a distance x from the origin at time t. The string is released from rest at time $t = 0$ with initial displacement $f(x) = \dfrac{1}{1+8x^2}$.

(a) Find the subsequent motion of this string $y(x,t)$.

The general solution to the wave equation with initial displacement $f(x)$ and initial velocity $g(x)$ is

$$y(x,t) = \frac{f(x+ct)+f(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(u)\,du$$

Release from rest $\Rightarrow$ $g(x) = 0$ at all times.

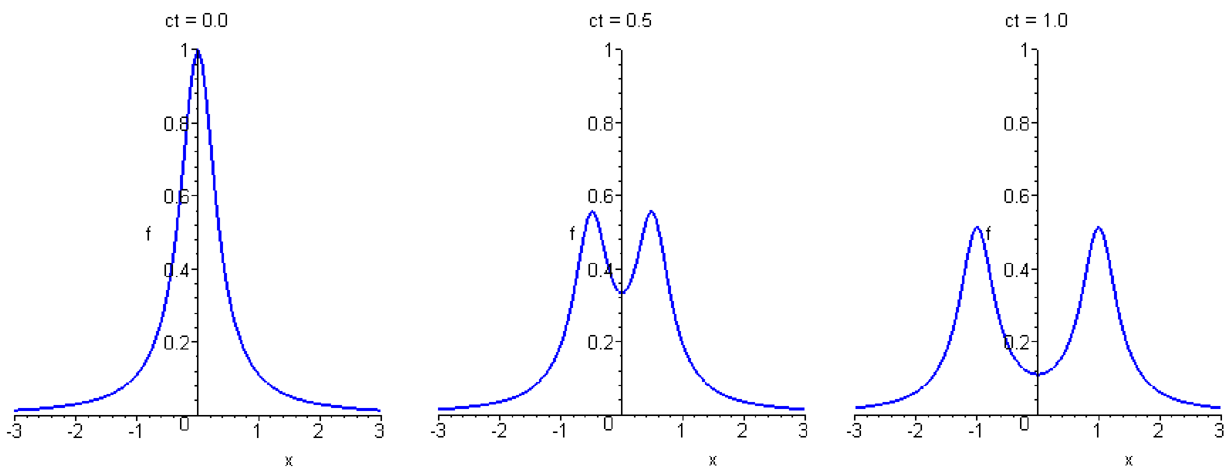
$$f(x) = \frac{1}{1+8x^2}.$$

The complete solution follows quickly:

$$\boxed{y(x,t) = \frac{1}{2}\left(\frac{1}{1+8(x+ct)^2} + \frac{1}{1+8(x-ct)^2}\right)}$$

(b) Sketch or plot the wave form at time $t = 0$ and at any two subsequent times.

There is an animation available from this Maple file.



4. Classify the partial differential equation $\nabla^2 u = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 12x$

and find the complete solution given the additional information

$u(0, y) = 0$, $\left.\left(\dfrac{\partial}{\partial x} u(x, y)\right)\right|_{x=0} = 3y^2$

$A = C = 1, \quad B = 0 \quad \Rightarrow \quad D = 0 - 4 = -4 < 0$

This PDE is therefore

elliptic everywhere

Solving the characteristic (or auxiliary) equation:

$$\lambda = \frac{-0 \pm \sqrt{-4}}{2} = \pm j$$

Complementary function:

$$u_C(x, y) = f\left(y + (-j)x\right) + g\left(y + jx\right),$$

Particular solution:

The right side is a first order polynomial.

The second derivatives of u_P must match that first order polynomial.

Therefore try

$$u_P = ax^3 + bx^2 y + cxy^2 + dy^3$$

$$\Rightarrow \quad (u_P)_x = 3ax^2 + 2bxy + cy^2 + 0, \quad (u_P)_y = 0 + bx^2 + 2cxy + 3dy^2$$

$$\Rightarrow \quad (u_P)_{xx} = 6ax + 2by + 0, \quad (u_P)_{yy} = 0 + 2cx + 6dy$$

Substitute into the PDE:

$$(u_P)_{xx} + (u_P)_{yy} = 6ax + 2by + 2cx + 6dy = 12x + 0y$$

Matching coefficients of x: $\quad 6a + 2c = 12 \quad \Rightarrow \quad c = 6 - 3a = 3(2 - a)$

Matching coefficients of y: $\quad 2b + 6d = 0 \quad \Rightarrow \quad b = -3d$

There are two free parameters (a and d). Leave them unresolved for now.

General solution:

$$u(x, y) = f\left(y + (-j)x\right) + g\left(y + jx\right) + ax^3 - 3dx^2 y + 3(2 - a)xy^2 + dy^3$$

$$\Rightarrow \quad u_x(x, y) = -j \cdot f'\left(y - jx\right) + j \cdot g'\left(y + jx\right) + 3ax^2 - 6dxy + 3(2 - a)y^2 + 0$$

4. (continued)

Imposing the boundary conditions,

$u(0, y) = 0 \quad \Rightarrow$

$f(y) + g(y) + 0 - 0 + 0 + dy^3 = 0$ **(A)**

$u_x(0, y) = 3y^2 \quad \Rightarrow$

$-j \cdot f'(y) + j \cdot g'(y) + 0 - 0 + 3(2-a)y^2 = 3y^2$ **(B)**

$\frac{d}{dy}(\mathbf{A})$: $\quad f'(y) + g'(y) + 3dy^2 = 0$ **(C)**

(B) $+ j$ **(C)** :

$0 + 2j \cdot g'(y) + 0 - 0 + 3(2 - a + d)y^2 = 3y^2 \quad \Rightarrow \quad g'(y) = \frac{3(d-a-1)y^2}{2j}$ **(D)**

(A) $\Rightarrow \quad f(y) = -g(y) - dy^3$ **(E)**

Now select values for a and d that simplify this problem as much as possible.
If we choose $a = 1$ and $d = 0$ then

(D) $\Rightarrow \quad g'(y) = 0 \quad \Rightarrow \quad g(y) = 0$ and

(E) $\Rightarrow \quad f(y) = 0$

so that the complete solution becomes

$u(x, y) = 0 + 0 + 1x^3 - 0x^2y + 3(2-1)xy^2 + 0y^3 \quad \Rightarrow$

$$\boxed{u(x, y) = x^3 + 3xy^2}$$

Note that the solution is ***not*** a harmonic function on $\mathbb{R}^2$, because $\nabla^2 u \neq 0$ (except on the y-axis).
However it is **subharmonic** on any domain entirely within the right half-plane ($x > 0$) and it is **superharmonic** on any domain entirely within the left half-plane.

5. Classify the partial differential equation $\dfrac{\partial^2 u}{\partial^2 x} - 7\dfrac{\partial^2 u}{\partial x\,\partial y} + 10\dfrac{\partial^2 u}{\partial^2 y} = -3$
and find the complete solution, given the additional information
$u(x,0) = 2x^2 + 4\,, \qquad u_y(x,0) = x$

$A = 1\,,\quad B = -7\,,\quad C = 10 \qquad \Rightarrow \quad D = 49 - 40 = 9 > 0$

This PDE is therefore

$$\boxed{\textbf{hyperbolic everywhere}}$$

Solving the characteristic (or auxiliary) equation:

$$\lambda = \frac{+7 \pm \sqrt{9}}{2} = 2 \text{ or } 5$$

The complementary function is therefore $u_C(x,y) = f(y+2x) + g(y+5x)$

The right side of the PDE is a constant and the left side includes only second derivatives.
Therefore try a pure second order polynomial function as the particular solution:

$$u_P = ax^2 + bxy + cy^2$$

$$\Rightarrow \quad (u_P)_x = 2ax + by + 0, \quad (u_P)_y = 0 + bx + 2cy$$

$$\Rightarrow \quad (u_P)_{xx} = 2a + 0 + 0, \quad (u_P)_{xy} = 0 + b + 0, \quad (u_P)_{yy} = 0 + 0 + 2c$$

$$\Rightarrow \quad (u_P)_{xx} - 7(u_P)_{xy} + 10(u_P)_{yy} = 2a - 7b + 20c = -3$$

There is only one constraint on three parameters.
Leave the choice of the two free parameters unresolved for now.

The general solution to the PDE is:

$$u(x,y) = f(y+2x) + g(y+5x) + ax^2 + bxy + cy^2 \quad (\text{where } 2a - 7b + 20c = -3)$$

$$\Rightarrow \quad \frac{\partial u}{\partial y} = f'(y+2x) + g'(y+5x) + 0 + bx + 2cy$$

Including the two additional items of information:

$$u(x,0) = 2x^2 + 4 \quad \Rightarrow \quad f(2x) + g(5x) + ax^2 + 0 + 0 = 2x^2 + 4 \qquad \textbf{(A)}$$

$$u_y(x,0) = x \quad \Rightarrow \quad f'(2x) + g'(5x) + bx + 0 = x \qquad \textbf{(B)}$$

$$\frac{d}{dx}(\textbf{A}): \quad 2f'(2x) + 5g'(5x) + 2ax = 4x \qquad \textbf{(C)}$$

$$\textbf{(C)} - 2\,\textbf{(B)}: \quad 0 + 3g'(5x) + (2a - 2b)x = (4-2)x \quad \Rightarrow \quad g'(5x) = \frac{2(1+b-a)x}{3} \qquad \textbf{(D)}$$

$$\textbf{(B)} \quad \Rightarrow \quad f'(2x) = (1-b)x - g'(5x) \qquad \textbf{(E)}$$

This problem is simplified if we choose $b = 1$ and $a = 2$. Then

5. (continued)

(D) $\Rightarrow \quad g'(5x) = 0 \quad \Rightarrow \quad g(5x) = 0$ and

(A) $\Rightarrow \quad f(2x) + 0 + 2x^2 = 2x^2 + 4 \quad \Rightarrow \quad f(2x) = 4 \quad \Rightarrow \quad f(x) = 4$

With $b = 1$ and $a = 2$ we have $4 - 7 + 20c = -3 \quad \Rightarrow \quad c = 0$

Therefore the complete solution is

$u(x, y) = 4 + 0 + 2x^2 + xy + 0 \quad \Rightarrow$

$$\boxed{u(x, y) = 2x^2 + xy + 4}$$

It is fairly straightforward to verify that this solution does satisfy the partial differential equation and both conditions:

$u(x, y) = 2x^2 + xy + 4 \quad \Rightarrow \quad u_x = 4x + y, \quad u_y = x$

$\Rightarrow \quad u_{xx} = 4, \quad u_{xy} = 1, \quad u_{yy} = 0$

$\Rightarrow \quad u_{xx} - 7u_{xy} + 10u_{yy} = 4 - 7 + 0 = -3$ ✓

$u(x, 0) = 2x^2 + 0 + 4$ ✓

$u_y(x, y) = x \quad \Rightarrow \quad u_y(x, 0) = x$ ✓

---

6. Classify the partial differential equation $\dfrac{\partial u}{\partial t} = 4\dfrac{\partial^2 u}{\partial x^2}$ and find its complete solution on the interval $0 \le x \le 100$ for all positive time *t*, given the additional information

$u(0, t) = 0$ and $u(100, t) = 100 \quad \forall t \ge 0$

and $u(x, 0) = 2x - \left(\dfrac{x}{10}\right)^2 \quad \forall x \in [0, 100]$

Also write down the steady state solution.

---

t is playing the role of y.

$A = 4, \quad B = 0, \quad C = 0 \quad \Rightarrow \quad D = 0 - 4(4)(0) = 0$

This is also the PDE for heat diffusion. Therefore

$$\boxed{\text{the PDE is } \textbf{parabolic}.}$$

6. (continued)

From page 8.36 of the lecture notes (Example 8.06.1), the complete solution of the heat equation (with the boundary conditions of constant temperatures T_1 and T_2 at the ends $x = 0$ and $x = L$ respectively and an initial temperature profile on $[0, L]$ of $u(x, 0) = f(x)$) is

$$u(x,t) = \left(\frac{T_2 - T_1}{L}\right)x + T_1 +$$

$$\frac{2}{L}\sum_{n=1}^{\infty}\left(\int_0^L \left(f(z) - \frac{T_2 - T_1}{L}z - T_1\right)\sin\left(\frac{n\pi z}{L}\right)dz\right)\sin\left(\frac{n\pi x}{L}\right)\exp\left(-\frac{n^2\pi^2 kt}{L^2}\right)$$

In this case $k = 4$, $L = 100$, $T_1 = 0$, $T_2 = 100$ and $f(x) = 2x - (x/10)^2$.

$$\Rightarrow \quad \frac{T_2 - T_1}{L}x + T_1 = \frac{100 - 0}{100}x + 0 = x$$

$$\Rightarrow \quad f(z) - \frac{T_2 - T_1}{L}z - T_1 = 2z - \left(\frac{z}{10}\right)^2 - z = z - \left(\frac{z}{10}\right)^2$$

The coefficients in the Fourier sine series become

$$\int_0^{100}\left(z - \left(\frac{z}{10}\right)^2\right)\sin\left(\frac{n\pi z}{100}\right)dz =$$

$$\left[-\left(\left(z - \left(\frac{z}{10}\right)^2\right)\frac{100}{n\pi} + \frac{1}{50}\left(\frac{100}{n\pi}\right)^3\right)\cos\left(\frac{n\pi z}{100}\right) + \left(1 - \frac{z}{50}\right)\left(\frac{100}{n\pi}\right)^2\sin\left(\frac{n\pi z}{100}\right)\right]_0^{100}$$

$$= \left(-\left(0 + \frac{1}{50}\left(\frac{100}{n\pi}\right)^3\right)\cos(n\pi) + 0\right)$$

$$- \left(-\left(0 + \frac{1}{50}\left(\frac{100}{n\pi}\right)^3\right) + 0\right) = \frac{1}{50}\left(\frac{100}{n\pi}\right)^3\left(1 - (-1)^n\right)$$

$$\Rightarrow \quad b_n = \begin{cases} \dfrac{1}{25}\left(\dfrac{100}{n\pi}\right)^3 & (n \text{ odd}) \\ 0 & (n \text{ even}) \end{cases}$$

Integration by parts:

$$\begin{array}{ccc} z - \left(\frac{z}{10}\right)^2 & & \sin\left(\frac{n\pi z}{100}\right) \\ & + & \\ 1 - \frac{z}{50} & & -\frac{100}{n\pi}\cos\left(\frac{n\pi z}{100}\right) \\ & - & \\ -\frac{1}{50} & & -\left(\frac{100}{n\pi}\right)^2\sin\left(\frac{n\pi z}{100}\right) \\ & + & \\ 0 & & \left(\frac{100}{n\pi}\right)^3\cos\left(\frac{n\pi z}{100}\right) \end{array}$$

6. (continued)

Also $\dfrac{2}{L} = \dfrac{1}{50}$. After some simplification of constants and substitution of $(2k-1)$ for the index of summation n, the complete solution becomes

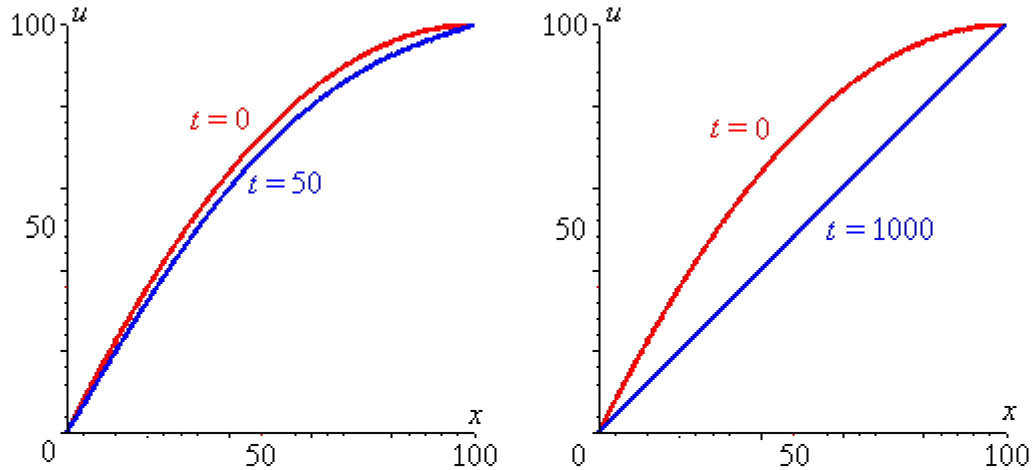
$$\boxed{u(x,t) = x + \frac{800}{\pi^3}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^3}\sin\left(\frac{(2k-1)\pi x}{100}\right)\exp\left(-\frac{(2k-1)^2\pi^2 t}{2500}\right)}$$

The Fourier series converges rapidly with increasing k.

The steady state solution is $\lim\limits_{t\to\infty} u(x,t) = x$

An animated version of the graph of this solution is available at
www.engr.mun.ca/~ggeorge/9420/solution/a7/q6graph.html

Two snapshots of $u(x, t)$:



[The Maple file used to generate these diagrams is available here.]

7. An ideal perfectly elastic string of length 1 m is fixed at both ends (at $x = 0$ and at $x = 1$). The string is displaced into the form $y(x,0) = f(x) = x^2(1-x)^2$ and is released from rest. Waves travel without friction along the string at a speed of 2 m/s. Find the displacement $y(x, t)$ at all locations on the string ($0 < x < 1$) and at all subsequent times ($t > 0$).

Write down the complete Fourier series solution and the first two non-zero terms.

From page 8.05 of the lecture notes, with $L = 1$ and $c = 2$, the complete solution can be quoted as

$$y(x,t) = \frac{2}{1}\sum_{n=1}^{\infty}\left(\int_0^1 f(u)\sin\left(\frac{n\pi u}{1}\right)du\right)\sin\left(\frac{n\pi x}{1}\right)\cos\left(\frac{2n\pi t}{1}\right)$$

$$c_n = \int_0^1 f(u)\sin(n\pi u)\,du = \int_0^1\left(u^2(1-u)^2\right)\sin(n\pi u)\,du$$

$$= \int_0^1\left(u^2-2u^3+u^4\right)\sin(n\pi u)\,du$$

$$= \left[\left(-\frac{\left(u^2-2u^3+u^4\right)}{n\pi}+\frac{2-12u+12u^2}{(n\pi)^3}-\frac{24}{(n\pi)^5}\right)\cos(n\pi u)\right.$$

$$\left.+\cancel{\left(\frac{2u-6u^2+4u^3}{(n\pi)^2}-\frac{-12+24u}{(n\pi)^4}\right)\sin(n\pi u)}\right]_0^1$$

$$= \left(-0+\frac{2}{(n\pi)^3}-\frac{24}{(n\pi)^5}\right)(-1)^n - \left(-0+\frac{2}{(n\pi)^3}-\frac{24}{(n\pi)^5}\right)$$

$u^2-2u^3+u^4$		$\sin(n\pi u)$
$2u-6u^2+4u^3$	$+$	$-\frac{\cos(n\pi u)}{n\pi}$
$2-12u+12u^2$	$-$	$-\frac{\sin(n\pi u)}{(n\pi)^2}$
$-12+24u$	$+$	$+\frac{\cos(n\pi u)}{(n\pi)^3}$
24	$-$	$+\frac{\sin(n\pi u)}{(n\pi)^4}$
0	$+$	$-\frac{\cos(n\pi u)}{(n\pi)^5}$

$$= \frac{2}{(n\pi)^3}\left(\frac{12}{(n\pi)^2}-1\right)\left(1-(-1)^n\right) = \begin{cases}\dfrac{4}{(n\pi)^3}\left(\dfrac{12}{(n\pi)^2}-1\right) & (n \text{ odd})\\[2ex] 0 & (n \text{ even})\end{cases}$$

Let $n = 2k-1, \quad (k \in \mathbb{N})$

7. (continued)

The complete solution is

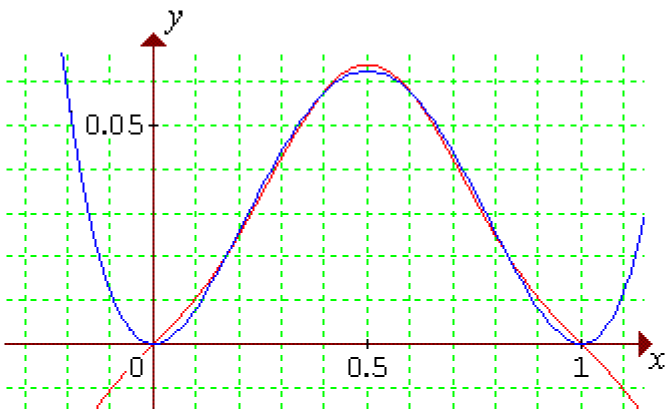
$$\boxed{y(x,t) = \frac{8}{\pi^3}\sum_{k=1}^{\infty}\left(\frac{1}{(2k-1)^3}\left(\frac{12}{((2k-1)\pi)^2}-1\right)\right)\sin((2k-1)\pi x)\cos(2(2k-1)\pi t)}$$

The series is also

$$y(x,t) = \frac{8}{\pi^3}\left(\left(\left(\frac{12}{\pi^2}-1\right)\right)\sin(\pi x)\cos(2\pi t) + \left(\frac{1}{27}\left(\frac{12}{(3\pi)^2}-1\right)\right)\sin(3\pi x)\cos(6\pi t) + \ldots\right)$$

Plotted here are $y(x,0) = f(x) = x^2(1-x)^2$ (in blue) on top of

$$\boxed{S_3(x) = \frac{8}{\pi^3}\left(\left(\left(\frac{12}{\pi^2}-1\right)\right)\sin(\pi x) + \left(\frac{1}{27}\left(\frac{12}{(3\pi)^2}-1\right)\right)\sin(3\pi x)\right)}$$ (in red)



The convergence of the Fourier series is so rapid that there is excellent agreement between the two plots, even when using just the first two non-trivial terms of the series!

A Maple animation of the wave is available at a7/q7graph.html.

☜ [Return to the index of assignments]()