When $\mathrm{P}[A \mid B] = \mathrm{P}[B \mid A]$

1. *Introduction*

Many students encountering probability theory for the first time have difficulty distinguishing conditional probabilities from joint or unconditional probabilities and they often confuse the conditional probabilities $\mathrm{P}[A \mid B]$ and $\mathrm{P}[B \mid A]$. For a pair of compatible events A, B whose unconditional probabilities are neither 0 nor 1, this note demonstrates two consequences when $\mathrm{P}[A \mid B] = \mathrm{P}[B \mid A]$: $\mathrm{P}[B] = \mathrm{P}[A]$ and $\mathrm{P}[A \mid \tilde{B}] = \mathrm{P}[B \mid \tilde{A}]$. The development of these consequences also provides some practice in the application of the laws of elementary probability.

2. $\mathrm{P}[B] = \mathrm{P}[A]$

The general multiplication law of probability quickly verifies that $\mathrm{P}[A \mid B]$ and $\mathrm{P}[B \mid A]$ are different, except when possible and compatible events A, B are equally likely:

$$\mathrm{P}[AB] = \mathrm{P}[B]\mathrm{P}[A \mid B] = \mathrm{P}[A]\mathrm{P}[B \mid A] \tag{1}$$

$$\Rightarrow \quad \mathrm{P}[A \mid B] = \frac{\mathrm{P}[A]\mathrm{P}[B \mid A]}{\mathrm{P}[B]} \tag{2}$$

If $\mathrm{P}[B] \neq \mathrm{P}[A]$ then $\mathrm{P}[A \mid B] \neq \mathrm{P}[B \mid A]$.

Among the serious consequences of a failure to distinguish between $\mathrm{P}[A \mid B]$ and $\mathrm{P}[B \mid A]$ is the now-notorious “prosecutor’s fallacy” [1]. One tragic case of a miscarriage of justice was summarised in the Mathematical Association President’s Address of 2003 [2]. In a criminal trial involving forensic evidence, if I represents the event that an accused person is innocent and M represents the event that a forensic match occurs, implicating the accused in the crime, then it is often the case that $\mathrm{P}[M \mid I]$ is tiny (much less than one in a thousand), but the jury needs to know $\mathrm{P}[I \mid M]$. From equation (2) they are connected by

$$\mathrm{P}[I \mid M] = \mathrm{P}[M \mid I] \cdot \frac{\mathrm{P}[I]}{\mathrm{P}[M]} \tag{3}$$

$\mathrm{P}[I \mid M]$ can be a substantially larger number, enough in some cases for $I \mid M$ to be odds on.

Equation (2) shows clearly that if $\mathrm{P}[A]$ and $\mathrm{P}[B]$ are non-zero and equal to each other, then $\mathrm{P}[A \mid B] = \mathrm{P}[B \mid A]$.

Rearranging equation (2) we have

$$\mathrm{P}[B] = \mathrm{P}[A] \cdot \frac{\mathrm{P}[B \mid A]}{\mathrm{P}[A \mid B]} \tag{4}$$

If events *A*, *B* are mutually exclusive then $\mathrm{P}[A \mid B] = \mathrm{P}[B \mid A] = 0$ and the expression for $\mathrm{P}[B]$ in equation (4) is indeterminate. $\mathrm{P}[A \mid B] = \mathrm{P}[B \mid A] \neq 0$ in equation (4) leads to $\mathrm{P}[B] = \mathrm{P}[A]$.

An appeal to symmetry between events *A*, *B* when $\mathrm{P}[A \mid B] = \mathrm{P}[B \mid A]$ also suggests that *A*, *B* should be equally likely, but this symmetry argument fails when the two events are mutually exclusive. The Venn probability diagram of figure 1 provides a simple counterexample.

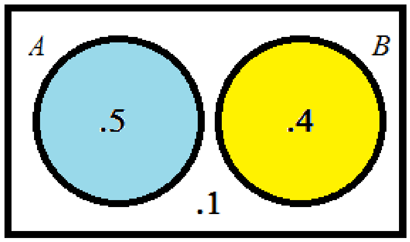

Figure 1: $\mathrm{P}[A \mid B] = \mathrm{P}[B \mid A]$ but $\mathrm{P}[B] \neq \mathrm{P}[A]$

3. $\mathrm{P}\left[A \mid \tilde{B}\right] = \mathrm{P}\left[B \mid \tilde{A}\right]$

Now we show that $\mathrm{P}[A \mid B] = \mathrm{P}[B \mid A] \neq 0$ forces $\mathrm{P}\left[A \mid \tilde{B}\right] = \mathrm{P}\left[B \mid \tilde{A}\right]$, (unless $\mathrm{P}[A] = \mathrm{P}[B] = 1$). From the definition of conditional probability (which follows from the general multiplication law of probability),

$$\mathrm{P}\left[A \mid \tilde{B}\right] = \frac{\mathrm{P}\left[A\tilde{B}\right]}{\mathrm{P}\left[\tilde{B}\right]} \tag{5}$$

Applying the general multiplication law of probability in the numerator,

$$\mathrm{P}\left[A \mid \tilde{B}\right] = \frac{\mathrm{P}[A]\,\mathrm{P}\left[\tilde{B} \mid A\right]}{\mathrm{P}\left[\tilde{B}\right]} \tag{6}$$

Applying the total probability law to pairs of complementary events,

$$\mathrm{P}\left[A \mid \tilde{B}\right] = \frac{\mathrm{P}[A]\left(1 - \mathrm{P}[B \mid A]\right)}{1 - \mathrm{P}[B]} \tag{7}$$

By a similar set of operations,

$$\mathrm{P}\left[B \mid \tilde{A}\right] = \frac{\mathrm{P}\left[B\tilde{A}\right]}{\mathrm{P}\left[\tilde{A}\right]} = \frac{\mathrm{P}[B]\,\mathrm{P}\left[\tilde{A} \mid B\right]}{\mathrm{P}\left[\tilde{A}\right]} = \frac{\mathrm{P}[B]\left(1 - \mathrm{P}[A \mid B]\right)}{1 - \mathrm{P}[A]} \tag{8}$$

But if $\mathrm{P}[A \mid B] = \mathrm{P}[B \mid A] \neq 0$ then $\mathrm{P}[B] = \mathrm{P}[A]$ and equations (7) and (8) both reduce to

$$\mathrm{P}\left[A \mid \tilde{B}\right] = \mathrm{P}\left[B \mid \tilde{A}\right] = \frac{\mathrm{P}[A]\left(1 - \mathrm{P}[B \mid A]\right)}{1 - \mathrm{P}[A]} \tag{9}$$

unless $\mathrm{P}[A] = \mathrm{P}[B] = 1$, in which case this expression for $\mathrm{P}\left[A \mid \tilde{B}\right]$ and $\mathrm{P}\left[B \mid \tilde{A}\right]$ is indeterminate (not surprising, when $\tilde{A}$, $\tilde{B}$ are both impossible events).

4. *Finding* $\mathrm{P}[A]$ *from* $\mathrm{P}[A \mid B]$ *and* $\mathrm{P}\left[A \mid \tilde{B}\right]$

When $\mathrm{P}[A \mid B] = \mathrm{P}[B \mid A] \neq 0$ and neither A nor B is the universal set, equation (9) leads to an expression for $\mathrm{P}[A]$ and $\mathrm{P}[B]$ in terms of $\mathrm{P}[A \mid B]$ and $\mathrm{P}\left[A \mid \tilde{B}\right]$ only.

$$\mathrm{P}\left[B \mid \tilde{A}\right] = \frac{\mathrm{P}[A]\left(1 - \mathrm{P}[B \mid A]\right)}{1 - \mathrm{P}[A]}$$

$$\Rightarrow \quad \left(1 - \mathrm{P}[A]\right)\mathrm{P}\left[B \mid \tilde{A}\right] = \mathrm{P}[A]\left(1 - \mathrm{P}[B \mid A]\right)$$

$$\Rightarrow \quad \mathrm{P}\left[B \mid \tilde{A}\right] = \mathrm{P}[A]\left(1 - \mathrm{P}[B \mid A] + \mathrm{P}\left[B \mid \tilde{A}\right]\right)$$

$$\Rightarrow \quad \mathrm{P}[A] = \frac{\mathrm{P}\left[B \mid \tilde{A}\right]}{1 - \mathrm{P}[B \mid A] + \mathrm{P}\left[B \mid \tilde{A}\right]} \tag{10}$$

or, equivalently,

$$\mathrm{P}[B] = \mathrm{P}[A] = \frac{\mathrm{P}\left[A \mid \tilde{B}\right]}{1 - \mathrm{P}[A \mid B] + \mathrm{P}\left[A \mid \tilde{B}\right]} \tag{11}$$

5. *Example*

Suppose that a current passes through a pair of pumping stations that are connected in parallel (as in figure 2). Each station has a 95% chance of operating properly if the other is functioning properly. However, a failure in one station puts more strain on the other station. The probability that either station operates properly when the other station has failed is only 20%. Find the unconditional probability for a station to operate properly and find the probability that the current will pass through this system.

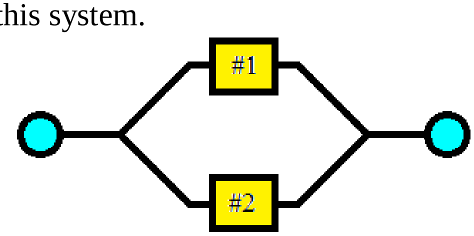

Figure 2: System connected in parallel

Solution

From the information in the question

$$\mathrm{P}[A \mid B] = \mathrm{P}[B \mid A] = .95 \quad \text{and} \quad \mathrm{P}\left[A \mid \tilde{B}\right] = \mathrm{P}\left[B \mid \tilde{A}\right] = .20$$

where A represents the event that pumping station #1 is functioning properly
and B represents the event that pumping station #2 is functioning properly

Equation (11) $\Rightarrow \quad \mathrm{P}[A] = \mathrm{P}[B] = \dfrac{.20}{1 - .95 + .20} = \dfrac{20}{25} = \dfrac{4}{5}$

The probability that a station is functioning is 80%, in the absence of knowledge about the status of the other station. That probability rises to 95% if it is known that the other station is working, but falls to 20% if it is known that the other station has failed. While they are identical, the two events A, B are strongly dependent.

Current will pass through the system if at least one of the stations is functioning.
The probability that current will pass through this system is

$$P[A \cup B] = P\left[\sim\left(\tilde{A} \cap \tilde{B}\right)\right] \quad \text{(deMorgan's law)}$$

$$= 1 - P\left[\tilde{A} \cap \tilde{B}\right] \quad \text{(complementary events)}$$

$$= 1 - P\left[\tilde{A}\right] P\left[\tilde{B} \mid \tilde{A}\right] \quad \text{(general multiplication law)}$$

$$= 1 - \left(1 - P[A]\right)\left(1 - P\left[B \mid \tilde{A}\right]\right) \quad \text{(complementary events)}$$

$$= 1 - \left(1 - \frac{4}{5}\right)\left(1 - \frac{1}{5}\right) = 1 - \frac{4}{25}$$

$$\Rightarrow \quad P[A \cup B] = \frac{21}{25} = 84\%$$

A direct approach is to partition the union into its three mutually exclusive and collectively exhaustive components:

$$P[A \cup B] = P[A \text{ only}] + P[B \text{ only}] + P[\text{both}]$$

$$= P\left[A \cap \tilde{B}\right] + P\left[\tilde{A} \cap B\right] + P[A \cap B]$$

But, from the total probability law, $P[A] = P\left[A \cap \tilde{B}\right] + P[A \cap B]$

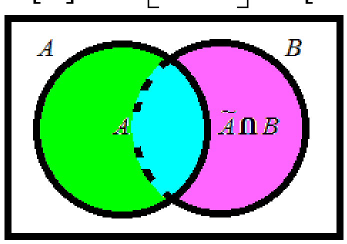

Figure 3: $P[A \cup B] = P[A] + P\left[\tilde{A} \cap B\right]$

$$\Rightarrow \quad P[A \cup B] = P[A] + P\left[\tilde{A} \cap B\right]$$

$$= P[A] + P\left[\tilde{A}\right] P\left[B \mid \tilde{A}\right] \quad \text{(general multiplication law)}$$

$$= .8 + .2 \times .20 = .80 + .04 = .84$$

Yet another approach is to use the general addition law of probability,

$$P[A \cup B] = P[A] + P[B] - P[A \cap B]$$

and then the general multiplication law of probability,

$$P[A \cup B] = P[A] + P[B] - P[A] P[B \mid A]$$

$$= .8 + .8 - .8 \times .95 = .8 \times 1.05 = .84$$

A tree diagram (figure 4) is a good visual method which illustrates the first two methods above for the calculation of $P[A \cup B]$.

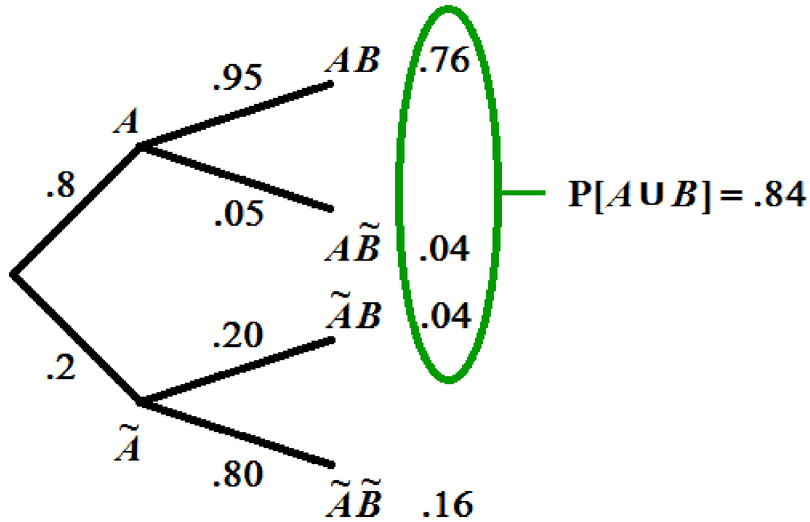

Figure 4: Tree diagram for $P[A \cup B]$

There is therefore a probability of 84% that current will pass through this system.

References
1. https://en.wikipedia.org/wiki/Prosecutor%27s_fallacy, accessed on 2016 Jan. 07.
2. B. Lewis, Taking Perspective (President's Address), *Math. Gaz.* **87** (November 2003), pp. 422-425.

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