Matching Socks

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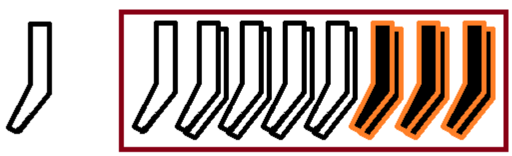
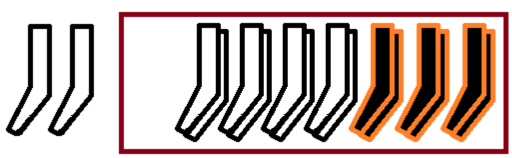
This note was inspired by "Quickie" problems 42A-C in the 2022 Autumn issue of "Symmetry plus" [1]. Two socks are pulled at random from a sock drawer containing some black socks and some white socks. We want the probability that the two socks drawn are the same colour to be ½ (matching socks are just as likely as a mismatch). How many pairs of each colour must be in the sock drawer for this to happen?

The solution with the smallest number of socks requires five pairs of one colour and three pairs of the other colour. Illustrated here is the case of a drawer of sixteen socks: ten white ($w = 5$) and six black ($b = 3$):

The probability that the first sock pulled from this sock drawer is white is $\mathrm{P}\left[W_1\right] = \frac{10}{16} = \frac{5}{8}$

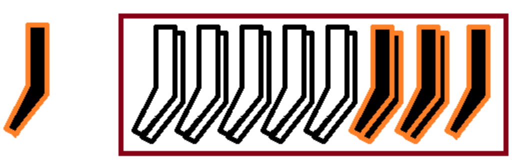


Given that the first sock is white, the drawer now contains nine white socks and all six black socks. The probability that the second sock pulled from this sock drawer is white, given that the first was white, is $\mathrm{P}\left[W_2 \mid W_1\right] = \frac{9}{15} = \frac{3}{5}$

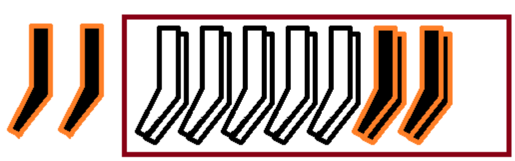


The joint probability that both socks are white is $\mathrm{P}\left[W_2 W_1\right] = \mathrm{P}\left[W_1\right] \times \mathrm{P}\left[W_2 \mid W_1\right] = \frac{5}{8} \cdot \frac{3}{5} = \frac{3}{8}$

The probability that the first sock pulled from this sock drawer is black is $\mathrm{P}\left[B_1\right] = \frac{6}{16} = \frac{3}{8}$



Given that the first sock is black, the drawer now contains five black socks and all ten white socks. The probability that the second sock pulled from this sock drawer is black, given that the first was black, is $\mathrm{P}\left[B_2 \mid B_1\right] = \frac{5}{15} = \frac{1}{3}$



The joint probability that both socks are black is $\mathrm{P}\left[B_2 B_1\right] = \mathrm{P}\left[B_1\right] \times \mathrm{P}\left[B_2 \mid B_1\right] = \frac{3}{8} \cdot \frac{1}{3} = \frac{1}{8}$

These two cases are mutually exclusive. Adding the two probabilities together yields the overall probability of matching socks: $\mathrm{P}[\text{match}] = \mathrm{P}[W_2 W_1] + \mathrm{P}[B_2 B_1] = \frac{3}{8} + \frac{1}{8} = \frac{1}{2}$

Now let's generalise.

Suppose that we have *b* pairs of black socks and *w* pairs of white socks in our sock drawer, for a total of 2*b* black socks and 2*w* white socks. We pull two socks at random from the drawer. The odds of a matching pair are known to be even (P[*E*] = 1/2). The number *b* of pairs of black socks is known. Find *w* in terms of *b*.

The total number of socks is 2*b* + 2*w*.
Following the method above,

$$\mathrm{P}[E] = \mathrm{P}[B_1 B_2] + \mathrm{P}[W_1 W_2]$$

$$= \mathrm{P}[B_1] \cdot \mathrm{P}[B_2 \mid B_1] + \mathrm{P}[W_1] \cdot \mathrm{P}[W_2 \mid W_1]$$

$$= \frac{2b}{2b+2w} \times \frac{2b-1}{2b+2w-1} + \frac{2w}{2b+2w} \times \frac{2w-1}{2b+2w-1}$$

$$\Rightarrow \quad \mathrm{P}[E] = \frac{2b(2b-1) + 2w(2w-1)}{(2b+2w) \times (2b+2w-1)}$$

But we require $\mathrm{P}[E] = \frac{1}{2}$. Clearing the denominators,

$$\Rightarrow \quad 2\big(2b(2b-1) + 2w(2w-1)\big) = (2b+2w)(2b+2w-1)$$

$$\Rightarrow \quad 8w^2 - 4w + \left(8b^2 - 4b\right) = 4w^2 + 8bw + 4b^2 - 2b - 2w$$

$$\Rightarrow \quad (8-4)w^2 + (-4+2-8b)w + \left(8b^2 - 4b - 4b^2 + 2b\right) = 0$$

$$\Rightarrow \quad 2w^2 - (4b+1)w + \left(2b^2 - b\right) = 0$$

Solving this quadratic equation for *w*,

$$w = \frac{4b + 1 \pm \sqrt{\left(16b^2 + 8b + 1\right) - \left(16b^2 - 8b\right)}}{4}$$

Simplifying,

$$w = b + \frac{1 \pm \sqrt{1+16b}}{4} \qquad (1)$$

For most values of *b* this solution leaves two invalid non-integer values for *w*. One of the two solutions for *w* is a whole number for only some values of *b*: 3, 5, 14, 18, 33, 39, 60, 68, 95, 105 etc. This formula is symmetric with respect to *b* and *w*:

$$b = w + \frac{1 \pm \sqrt{1+16w}}{4}$$

Which square root to take (positive or negative) depends on which of *b* and *w* is greater.

The solutions therefore occur in pairs:
$(b, w) = (3, 5), (5, 3), (14, 18), (18, 14)$, etc.
One can construct a spreadsheet to search for the first few integer solutions (b, w).

In problem 42A of [1], $b = w + 4$. Either by substitution into (1) or by inspection of the list above, the solution is $b = 18$, $w = 14$. The total number of socks in this case is $2b + 2w = 64$.

In problem 42B, $b = 33$. It soon follows that $w = 39$, for a total of $2b + 2w = 144$ socks.

In problem 42C, the information provided is $b > w$ and $2b + 2w = 400$. We can substitute $w = 200 - b$ into (1) or we can search through the pairs of integer solutions for the case $b + w = 200$, where we find the solution $(b, w) = (105, 95)$ and hence 210 black socks.

Another pattern that becomes apparent is that integer solutions, (for the numbers of pairs of socks of each colour for which $P[E] = 1/2$), are possible only when the total number of socks is a square natural number multiple of 16: $2b + 2w = 16n^2$.

$P[E] = 1/2$ is possible only when the total number of socks is one of 16, 64, 144, 256, 400, etc. Why?

Returning to equation (1), an integer value for the number w of pairs of white socks requires that

$\frac{1 \pm \sqrt{1+16b}}{4}$ be an integer – call it n.

Solving for b in terms of n,

$$\frac{1 \pm \sqrt{1+16b}}{4} = n \quad \Rightarrow \quad \pm\sqrt{1+16b} = 4n - 1$$

$$\Rightarrow \quad 1 + 16b = (4n-1)^2$$

$$\Rightarrow \quad 16b = (4n-1)^2 - 1 = \big((4n-1)-1\big)\big((4n-1)+1\big)$$

$$\Rightarrow \quad b = \frac{(4n-2)4n}{16} = \frac{n(2n-1)}{2} = \frac{n}{2} \times (\text{an odd number})$$

This expression for b is an integer if and only if n is even. Let $n = 2k$, then

$$b = k(4k-1) \tag{2}$$

and, from (1),

$$w = b + 2k = 4k^2 - k + 2k = 4k^2 + k$$

so that $w = k(4k+1)$ (3)

The total number of socks is then

$$2b + 2w = 2k\big((4k-1) + (4k+1)\big) = 16k^2,$$

where k can be any natural number.
Again, from symmetry, the values of b and w can be interchanged.

The first few solutions are

k	(b, w) or (w, b)	number of pairs $(b+w)$	number of socks
1	(3, 5)	8	16
2	(14, 18)	32	64
3	(33, 39)	72	144
4	(60, 68)	128	256
5	(95, 105)	200	400
6	(138, 150)	288	576
7	(189, 203)	392	784

Another pattern that is easy to detect and to verify is
$| b - w | = 2k$

As the number of socks contained in our sock drawer becomes unreasonably large, the ratio of the numbers of socks of each colour needed to keep P[E] = 1/2 clearly approaches 1:
from (2) and (3),

$$\frac{\max(b,w)}{\min(b,w)} = \frac{k(4k+1)}{k(4k-1)} = \frac{4 + 1/k}{4 - 1/k} \to 1 \text{ as } k \to \infty$$

An Excel spreadsheet also reveals an interesting pattern in the ratio of the probabilities of drawing two white socks to drawing two black socks. For the case with the least possible numbers of pairs of socks (five pairs of white socks and three pairs of black socks, or vice versa), this ratio is 3:1. More generally (for $w > b$), the spreadsheet suggests

$$\frac{\mathrm{P}[WW]}{\mathrm{P}[BB]} = 1 + \frac{2}{2k-1} \text{ (which clearly approaches 1 as } k \to \infty\text{)}$$

Verifying this observed pattern,

$$\frac{\mathrm{P}[WW]}{\mathrm{P}[BB]} = \frac{2w(2w-1)}{(2b+2w)\times(2b+2w-1)} \times \frac{(2b+2w)\times(2b+2w-1)}{2b(2b-1)}$$

$$= \frac{w(2w-1)}{b(2b-1)}$$

From (2) and (3),

$$\frac{\mathrm{P}[WW]}{\mathrm{P}[BB]} = \frac{k(4k+1)\big(2k(4k+1)-1\big)}{k(4k-1)\big(2k(4k-1)-1\big)}$$

$$= \frac{(4k+1)(8k^2+2k-1)}{(4k-1)(8k^2-2k-1)}$$

$$= \frac{(4k-1+2)\left(8k^2-2k-1+4k\right)}{(4k-1)\left(8k^2-2k-1\right)}$$

The numerator is

$$(4k-1)\left(8k^2-2k-1\right)+(4k-1)4k \;+\; 2\left(8k^2+2k-1\right)$$

$$= (4k-1)\left(8k^2-2k-1\right)+2\left(8k^2-2k\right) \;+\; 2\left(8k^2+2k-1\right)$$

$$= (4k-1)\left(8k^2-2k-1\right)+2\left(16k^2-1\right)$$

The factors in the denominator are $(4k-1)(4k+1)(2k-1)$

The ratio becomes

$$1 + \frac{2(4k-1)(4k+1)}{(4k-1)(4k+1)(2k-1)} = 1 + \frac{2}{2k-1}$$

Obvious extensions of this work are for

- values for probability of matching socks other than 1/2;
- more than two colours of socks in the drawer.

Reference

1. "Quickie problems 42", *Symmetry plus*, 2022 Autumn, No. 79, pages 2, 5 and 7

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