The Integral of $\frac{1}{x}$

by G.H. George

Students meeting the result $\int \frac{1}{x}\,dx = \ln x + C$ for the first time are often amazed by the fact that a function so unlike x^{n+1} can "fill the gap" at $n = -1$ in the integration of x^n. Here we show that $\ln x$ must fill that gap, by examination of the limit of $\int x^n\,dx$ as $n \to -1$. Consider the function defined by

$$I(n,b) = \int_1^b t^n\,dt\,, \quad n \neq -1$$

where b is a positive constant.

The function $I(n, b)$ has a discontinuity at $n = -1$, but everywhere else $I(n,b) = \dfrac{b^{n+1} - 1}{n+1}$ is continuous.

Evaluate a simple Maclaurin series expansion of $I(n, b)$ in x, with $b = 1 + x$:

$$I(n,1+x) = \frac{(1+x)^{n+1} - 1}{n+1} =$$

$$\frac{1 + \dfrac{(n+1)}{1}x + \dfrac{(n+1)n}{2\times 1}x^2 + \dfrac{(n+1)n(n-1)}{3\times 2\times 1}x^3 + \ldots - 1}{n+1}$$

$$\Rightarrow \quad I(n,1+x) = x + \frac{n}{2\times 1}x^2 + \frac{n(n-1)}{3\times 2\times 1}x^3 + \frac{n(n-1)(n-2)}{4\times 3\times 2\times 1}x^4 + \ldots$$

$$\Rightarrow \quad \lim_{n\to -1} I(n,1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \quad = \ln(1+x) = \ln b$$

An alternative is to use l'Hôpital's rule:

$$\lim_{n\to -1} I(n,b) = \lim_{n\to -1} \frac{b^{n+1} - 1}{n+1} \overset{H}{=} \lim_{n\to -1} \frac{b^{n+1}\ln b - 0}{1} = \ln b$$

The discontinuity may therefore be removed by redefining $I(n, b)$ to be

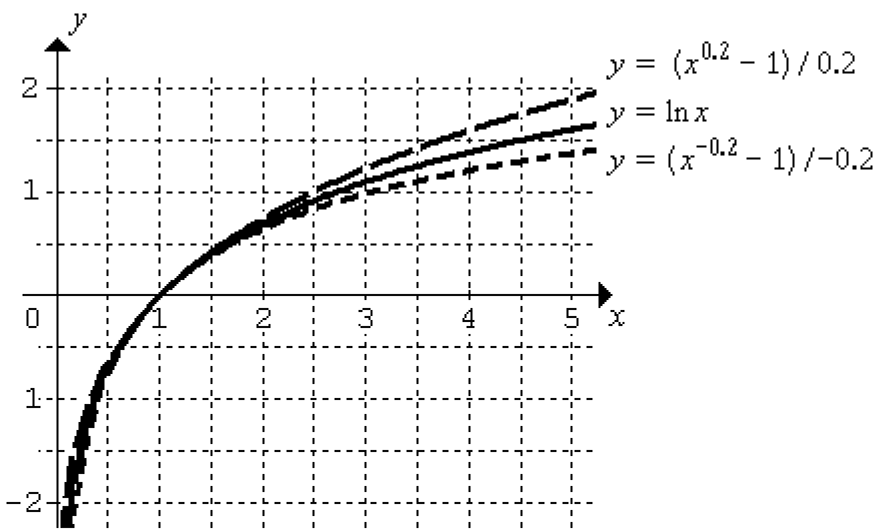
$$I(n,b) = \begin{cases} \dfrac{b^{n+1} - 1}{n+1} & (n \neq -1) \\ \ln b & (n = -1) \end{cases}$$

But $I(n, b)$ was also defined to be $I(n,b) = \int_1^b x^n\,dx$.

It then follows that $\int_1^b x^n\,dx \to [\ln x]_1^b$ as $n \to -1$ and $\ln x$ does indeed fill the gap.

A graph of $y(x) = \begin{cases} \dfrac{x^{n+1}-1}{n+1} & (n \neq -1) \\ \ln x & (n = -1) \end{cases}$ against x for three values of n,

(n = –1.2, –1 and –0.8), illustrates this limiting behaviour:



I am grateful to an anonymous referee for suggestions that have improved this note and for the following extension to this work.

$$\lim_{n\to -1} \frac{x^{n+1}-1}{n+1} = \ln x \qquad \Rightarrow \qquad \lim_{m\to 0} \frac{x^m - 1}{m} = \ln x$$

$$\Rightarrow \quad \lim_{N\to\infty} N\left(x^{1/N} - 1\right) = \ln x$$

This can be used as the basis of a method for estimating natural logarithms on a basic calculator, using just the square root key and the three arithmetic keys –, ×, = :

$$2^a\left(\left(\sqrt{x}\right)^a - 1\right) \to \ln x \quad \text{as} \quad a \to \infty$$

For example, to estimate ln 3 (with a = 5), press ‘3’, then press the square root key five times to obtain $3^{1/2^5} = 3^{1/32} = 1.034\,927\,767\ldots$, subtract 1, then double five times to obtain $1.117\,688\ldots$, which is a mediocre estimate of $\ln 3 = 1.098\,612\,288\ldots$.

Increasing the number of repeated key presses improves the estimate up to a certain point, determined by the level of precision to which floating point numbers are stored in the calculator. Entering '3' and pressing the square root key ten times on a good modern calculator produces $3^{1/2^{10}} = 3^{1/1024} = 1.001073439\ldots$. Subtract 1, then double ten times to obtain $1.099201830\ldots$, which is $\ln 3$ correct to four significant figures. I still use a Casio fx-120 calculator from 1978, which possesses a much lower level of precision. Application of this procedure (with $a = 10$) on my old calculator produces nearly the same estimate, 1.099200512.

The dependence on the level of precision becomes apparent by $a = 20$: From a good modern calculator, the estimate of $\ln 3$ is $1.098612864\ldots$ (which is correct to eight significant figures), whereas the estimate from my Casio fx-120: is 1.096810496, (which is less accurate than the $a = 10$ case).

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