When copycats lose

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This note was inspired by a puzzle in the *Guardian Weekly* newspaper [1, puzzle 3].

Two people take part in a game where each in turn flips a pair of fair coins. There are two results of interest in each turn: either a player's coins are both heads or they are not. If one player's coins are both heads then the other player, to avoid defeat, must get at least one tail. If one player's coins are not both heads, then the other player must get double heads. The game keeps going, with the two players alternating double coin flips, until one of them loses by repeating the other's most recent result (double-heads or not). Assume that all coin flips are independent of each other.

Figure 1 provides an example of a game won by player 2 on player 1's third turn:

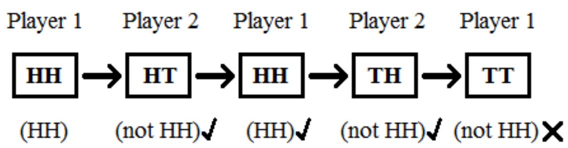

FIGURE 1: sample game. Player 2 wins because Player 1 has copied (not HH).

Two different methods are presented here for evaluating the probability that the player who goes first wins. Conditional probabilities for eventual victory following the first turn are also developed.

Infinite series method

Let the event A = both coins in the flip of two coins are heads, then $\text{P}[A] = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$. Let $p = \text{P}[A] = \frac{1}{4}$ and $q = \text{P}[\tilde{A}] = 1 - p = \frac{3}{4}$. Player 1 wins in the first round if player 2's initial outcome matches player 1's:

$$\text{P}[\text{player 1 wins in the first round}] = \text{P}[AA \cup \tilde{A}\tilde{A}] = \text{P}[AA] + \text{P}[\tilde{A}\tilde{A}]$$

$$= (\text{P}[A])^2 + (\text{P}[\tilde{A}])^2 = p^2 + q^2 = \left(\frac{1}{4}\right)^2 + \left(\frac{3}{4}\right)^2 = \frac{1}{16} + \frac{9}{16} = \frac{10}{16} = \frac{5}{8}.$$

The corresponding odds are $\dfrac{\frac{5}{8}}{1 - \frac{5}{8}} = \dfrac{5}{8 - 5} = 5 : 3$ on.

The game does not end in round 1 if $A\tilde{A}$ or $\tilde{A}A$ occurs.

Player 1 wins if, and only if, an event in either of the following sequences occurs:

$$AA,\ A\tilde{A}AA,\ A\tilde{A}A\tilde{A}AA,\ A\tilde{A}A\tilde{A}A\tilde{A}AA,\ \ldots$$

or

$$\tilde{A}\tilde{A},\ \tilde{A}A\tilde{A}\tilde{A},\ \tilde{A}A\tilde{A}A\tilde{A}\tilde{A},\ \tilde{A}A\tilde{A}A\tilde{A}A\tilde{A}\tilde{A},\ \ldots .$$

The probability of a win by the first sequence is

$$p_1 = pp + pqpp + pqpqpp + pqpqpqpp + \ldots = p^2\left(1 + pq + (pq)^2 + (pq)^3 + \ldots\right)$$

which is the sum of a geometric series of first term $a = p^2 = \dfrac{1}{16}$

and common ratio $r = pq = \dfrac{1}{4} \times \dfrac{3}{4} = \dfrac{3}{16}$.

Because $|r| < 1$, the series converges absolutely to the sum

$$p_1 = \frac{a}{1 - r} = \frac{p^2}{1 - pq} = \frac{\frac{1}{16}}{1 - \frac{3}{16}} = \frac{1}{16 - 3} = \frac{1}{13}.$$

Similarly, the probability of a win by the second sequence is

$$\begin{aligned} p_2 &= qq + qpqq + qpqpqq + qpqpqpqq + \ldots \\ &= q^2\left(1 + pq + (pq)^2 + (pq)^3 + \ldots\right) = \frac{q^2}{1 - pq} \end{aligned}$$

$$\Rightarrow p_2 = \frac{\frac{9}{16}}{1 - \frac{3}{16}} = \frac{9}{16 - 3} = \frac{9}{13}.$$

The two sequences are mutually exclusive

$$\Rightarrow \mathrm{P}\left[\text{player 1 wins}\right] = p_1 + p_2 = \frac{1 + 9}{13} = \frac{10}{13}.$$

The corresponding odds are $\dfrac{10}{13 - 10} = 10 : 3$ on.

For the original puzzle $p = \frac{1}{4}$. The analysis is valid for similar ‘copycat’ scenarios with any value of p in the range $0 \leqslant p \leqslant 1$.

Recursion method

In one round of play (after both players have had one turn) there are four possible outcomes:

AA : player 1 wins immediately (probability p^2)
$A\tilde{A}$: the game continues (probability pq)
$\tilde{A}A$: the game continues (probability qp)
$\tilde{A}\tilde{A}$: player 1 wins immediately (probability q^2)

The probability that player 1 wins on player 2's first turn is $p^2 + q^2$.

The minimum value of this probability for all valid values of p $(0 \leqslant p \leqslant 1)$ is determined easily:

$$f(p) = p^2 + (1 - p)^2 = 2p^2 - 2p + 1 = 2\left(p - \tfrac{1}{2}\right)^2 + \tfrac{1}{2} \Rightarrow f(p) \geqslant \tfrac{1}{2}.$$

with equality (and the absolute minimum value of $\frac{1}{2}$) occurring at $p = \frac{1}{2}$.

The absolute maximum value of 1 occurs at the endpoints of the domain $p = 0$ and $p = 1$.

Unless $\text{P}[A] = \frac{1}{2}$ it is odds on that player 1 will win in the first round of play.

Let us now explore what happens if the game does not end on player 2's first turn.

Scenario $A\tilde{A}$:

Let $p_3 = \text{P}[\text{player 1 wins}|A\tilde{A}]$ then the possibilities in the next round are

AA : player 1 wins in this round (probability p^2)

$A\tilde{A}$: the game continues (probability pq)

$\tilde{A}$: player 1 loses in this round (probability q)

If the game continues, then we are back at the same scenario again, with a probability at that point of p_3 that player 1 wins eventually. Therefore

$$p_3 = p^2 + pq \times p_3 + 0 \Rightarrow (1 - pq)p_3 = p^2 \Rightarrow p_3 = \frac{p^2}{1 - pq}.$$

Scenario $\tilde{A}A$:

Let $p_4 = \text{P}[\text{player 1 wins}|\tilde{A}A]$ then the possibilities in the next round are

A : player 1 loses in this round (probability p)

$\tilde{A}A$: the game continues (probability qp)

$\tilde{A}\tilde{A}$: player 1 wins in this round (probability q^2)

If the game continues, then we are back at the same scenario again, with a probability at that point of p_4 that player 1 wins eventually. Therefore

$$p_4 = 0 + qp \times p_4 + q^2 \Rightarrow (1 - pq)p_4 = q^2 \Rightarrow p_4 = \frac{q^2}{1 - pq}.$$

Therefore the overall probability that player 1 wins this game is

$$p^2 + q^2 + pqp_3 + qpp_4 = p^2 + q^2 + \left(\frac{p^2}{1 - pq} + \frac{q^2}{1 - pq}\right)$$

$$= (p^2 + q^2)\left(1 + \frac{pq}{1 - pq}\right) = (p^2 + q^2)\left(\frac{1 - pq + pq}{1 - pq}\right)$$

$$\Rightarrow \text{P}[\text{player 1 wins}] = \frac{p^2 + q^2}{1 - pq}$$

and the corresponding odds are $r = \dfrac{p^2 + q^2}{(1 - pq) - (p^2 + q^2)}$.

For the double coin flip example above $p = \frac{1}{4}$. Substituting into these general expressions we recover

$$\text{P}\left[\text{player 1 wins in first round}\right] = \left(\frac{1}{4}\right)^2 + \left(\frac{3}{4}\right)^2 = \frac{10}{16} = \frac{5}{8}$$

with corresponding odds of $5 : 3$ on and similarly $\text{P}\left[\text{player 1 wins}\right] = \dfrac{10}{13}$ with odds of $10 : 3$ on.

Figure 2 shows the graph of the probabilities of both (player 1 wins in round 1) and (player 1 wins) as functions of $p = \text{P}[A]$.

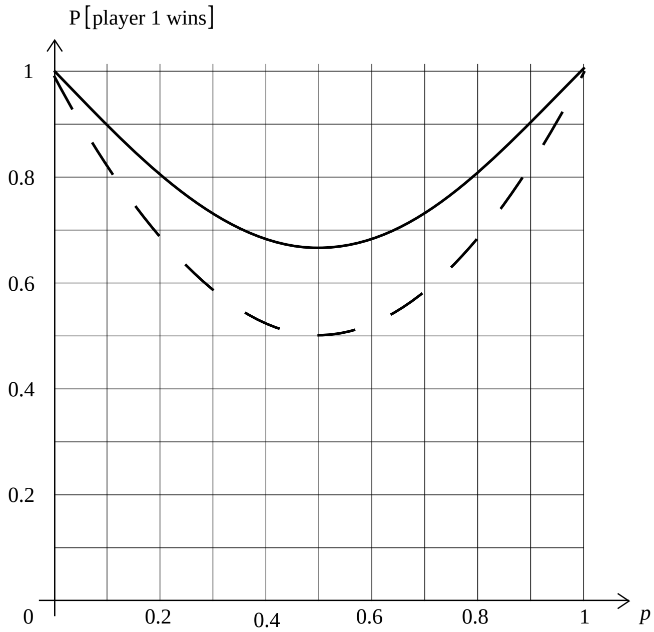

FIGURE 2: Graphs of $y = p^2 + (1 - p)^2$ [dashed] and $y = \dfrac{p^2 + (1 - p)^2}{1 - p(1 - p)}$ [solid]

Both the probability that player 1 wins in the first round and the [greater] probability that player 1 wins are at their lowest when $\text{P}[A] = p = \frac{1}{2}$.

An example of this worst case scenario for player 1 is

$A =$ even number on one roll of a fair die. $p = \frac{1}{2}, q = \frac{1}{2} \Rightarrow pq = \frac{1}{4}$

The probability that player 1 wins on player 2's first turn is $p^2 + q^2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$.

The probability that player 1 wins (sooner or later) is

$$\frac{p^2 + q^2}{1 - pq} = \frac{1}{2} \times \frac{1}{1 - \frac{1}{4}} = \frac{2}{3}.$$

At first sight one might have expected the probability for Player 1 to win to be a monotonic function of p. However, the game is based on copying versus not copying. The probability of winning should be the same if we swap the probabilities p and $q = 1 - p$. We therefore obtain a function that is symmetric about $p = \frac{1}{2}$.

The odds on player 1 winning the game are never worse than 2:1 on, as Figure 2 illustrates. It may seem counter-intuitive that the game is not fair and that Player 1 has such a strong advantage. Players win by not copying, which gives Player 1 an automatic advantage. Player 1 starts the game and therefore cannot have copied on Player 1's first turn.

After player 1's initial turn

The situation immediately after the first player's initial turn is somewhat different.

Suppose that event A has happened for player 1. Then two outcomes are possible for the end of the first round:

AA : player 1 wins immediately (probability p)

$A\tilde{A}$: the game continues (probability q)

In the latter case, we have seen above that the probability of eventual victory for player 1 is $p_3 = \frac{p^2}{1 - pq}$. Therefore

$$\mathrm{P}\left[\text{player 1 wins} | A\right] = p + \frac{qp^2}{1 - pq} = \frac{p(1 - pq + pq)}{1 - pq} = \frac{p}{1 - pq}.$$

Suppose that event $\tilde{A}$ has happened for player 1. Then two outcomes are possible for the end of the first round:

$\tilde{A}\tilde{A}$: player 1 wins immediately (probability q)

$\tilde{A}A$: the game continues (probability p)

In the latter case, we have seen above that the probability of eventual victory for player 1 is $p_4 = \frac{q^2}{1 - pq}$. Therefore

$$\mathrm{P}\left[\text{player 1 wins} | \tilde{A}\right] = q + \frac{q^2 p}{1 - pq} = \frac{q(1 - pq + pq)}{1 - pq} = \frac{q}{1 - pq}.$$

Combining these cases, using the law of total probability (the Partition Theorem, see [2, p. 14]), we recover the overall probability that player 1 wins:

$$\begin{aligned}\mathrm{P}\left[\text{player 1 wins}\right] &= \mathrm{P}\left[\text{player 1 wins} \cap A\right] + \mathrm{P}\left[\text{player 1 wins} \cap \tilde{A}\right] \\ &= \mathrm{P}\left[A\right] \cdot \mathrm{P}\left[\text{player 1 wins} | A\right] + \mathrm{P}\left[\tilde{A}\right] \cdot \mathrm{P}\left[\text{player 1 wins} | \tilde{A}\right] \\ &= p.\frac{p}{1 - pq} + q.\frac{q}{1 - pq} = \frac{p^2 + q^2}{1 - pq}.\end{aligned}$$

Examples:

1. Double coin toss. A = both coins are heads. $p = \frac{1}{4}, q = \frac{3}{4} \Rightarrow pq = \frac{3}{16}$.

$$\mathrm{P}\left[\text{player 1 wins}|A\right] = \frac{p}{1-pq} = \frac{\frac{1}{4}}{1-\frac{3}{16}} = \frac{4}{16-3} = \frac{4}{13} \text{ or 9:4 against.}$$

$$\mathrm{P}\left[\text{player 1 wins}|\tilde{A}\right] = \frac{q}{1-pq} = \frac{\frac{3}{4}}{1-\frac{3}{16}} = \frac{12}{16-3} = \frac{12}{13} \text{ or 12:1 on.}$$

2. A = even number on one fair die. $p = \frac{1}{2}, q = \frac{1}{2}, \Rightarrow pq = \frac{1}{4}$.

$$\mathrm{P}\left[\text{player 1 wins}|A\right] = \frac{p}{1-pq} = \frac{\frac{1}{2}}{1-\frac{1}{4}} = \frac{2}{4-1} = \frac{2}{3} \text{ or 2:1 on.}$$

$$\mathrm{P}\left[\text{player 1 wins}|\tilde{A}\right] = \frac{q}{1-pq} = \frac{\frac{1}{2}}{1-\frac{1}{4}} = \frac{2}{4-1} = \frac{2}{3} \text{ or 2:1 on.}$$

Figure 3 illustrates these conditional probabilities as functions of $p = \mathrm{P}[A]$.

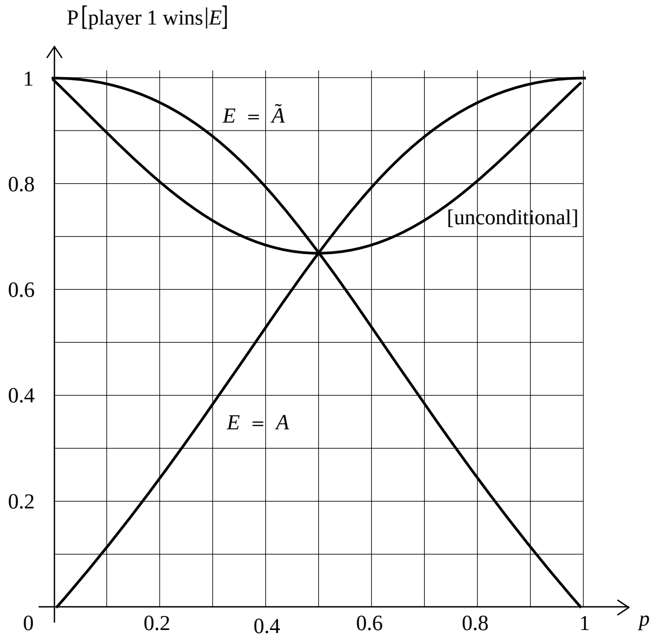

FIGURE 3: Graphs of $y = \dfrac{p}{1-p(1-p)}$, $y = \dfrac{1-p}{1-p(1-p)}$ and $y = \dfrac{p^2+(1-p)^2}{1-p(1-p)}$

When $\mathrm{P}[A] = \frac{1}{2}$, player 1 has odds of 2:1 on to win no matter what happens on his first turn. Otherwise, those odds are even better if the more likely of A and $\tilde{A}$ happens on his first turn.

Even if the less likely event happens, the odds remain in favour of player 1 unless

$$\frac{p}{1 - pq} < \frac{1}{2} \Rightarrow 2p < 1 - p(1 - p) \Rightarrow p^2 - 3p + 1 > 0 \Rightarrow \left(\tfrac{3}{2} - p\right)^2 + 1 - \tfrac{9}{4} > 0$$

$$\Rightarrow \left(\tfrac{3}{2} - p\right)^2 > \tfrac{5}{4} \Rightarrow \frac{3}{2} - p > \frac{\sqrt{5}}{2} \Rightarrow p < \frac{3 - \sqrt{5}}{2} \approx 0.382$$

or

$\frac{q}{1 - pq} < \frac{1}{2}$, which, by symmetry, leads to $q = 1 - p < \frac{3 - \sqrt{5}}{2}$ $\Rightarrow p > 1 - \frac{3 - \sqrt{5}}{2} \approx 0.618$.

It is interesting to note the relationship with the golden ratio $\phi = \frac{1}{2}\left(1 + \sqrt{5}\right)$: the critical values of p are also $2 - \phi = 1 - \frac{1}{\phi} \approx 0.382$ and $\phi - 1 = \frac{1}{\phi} \approx 0.618$.

Therefore the odds remain in favour of player 1 regardless of the outcome of player 1's first turn, provided that $0.382 \leqslant p \leqslant 0.618$ or, more precisely,

$$\frac{3 - \sqrt{5}}{2} < p < \frac{\sqrt{5} - 1}{2}.$$

Acknowledgement

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Reference

1. Chris Maslanka, Maslanka puzzles, *Guardian Weekly*, **191** (8) (1 August 2014) p. 44.
2. G. Grimmett and D. Welsh, *Probability: an introduction*, (2nd edn.) Oxford University Press, Oxford.

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