Parallel probabilities

GLYN GEORGE

Introduction

After several years of teaching an introduction to probability and statistics for engineering degree students, my attention has been captured by some variations on the familiar general addition law of probability. Network analysis of components connected in parallel is one of many applications.

To evaluate $\mathrm{P}\left[A \cup B\right]$, it is sufficient to have values for $\mathrm{P}[A]$, $\mathrm{P}[B]$ and the conditional probability $\mathrm{P}[A \mid B]$ (or $\mathrm{P}[B \mid A]$). The evaluation is very easy:

$$\mathrm{P}[A \cup B] = \mathrm{P}[A] + \mathrm{P}[A] - \mathrm{P}[A \cap B] = \mathrm{P}[A] + \mathrm{P}[B] - \mathrm{P}[A \mid B] \times \mathrm{P}[B].$$

There may be situations where the only direct information involves three of the conditional probabilities $\mathrm{P}[A \mid B]$, $\mathrm{P}[A \mid \tilde{B}]$, $\mathrm{P}[B \mid A]$, $\mathrm{P}[B \mid \tilde{A}]$ (or their complements). Under those circumstances, the calculation $\mathrm{P}[A \cup B]$ of requires more thought and provided motivation for this Article.

In a simple analysis of network reliability, components can be connected in series or in parallel. Examples of networks are the transmission of a signal or an electric current between two points, and the flow of fluid through pumping stations from one point to another.

We shall take the binary case; either a component is working normally or it has failed completely. A further simplification is to consider identical components only.

Connection in series

All components that are connected in series must work in order for that subsystem to work.

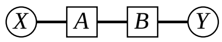

FIGURE 1: Two components connected in series

Let $\mathrm{P}[E]$ represent the (unconditional) probability that component E is working. With identical components, $\mathrm{P}[A] = \mathrm{P}[B]$. Let this common probability be represented by p. The probability of successful transmission for these two components connected in series is $\mathrm{P}[A \cap B]$. By the general multiplication law of probability,

$$\mathrm{P}[A \cap B] = \mathrm{P}[A] \times \mathrm{P}[B \mid A].$$

It is reasonable to assume that the reliabilities of identical components connected in series are independent. Then the probability of successful transmission is

$$\mathrm{P}[A \cap B] = \mathrm{P}[A] \times \mathrm{P}[A] = p^2.$$

Connection in parallel

Only one component in a set connected in parallel needs to work in order for the subsystem to work.

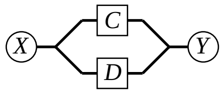

FIGURE 2: Two components connected in parallel

The probability of successful transmission for these two components connected in parallel is $\mathrm{P}[C \cup D]$.

By the general addition law of probability,

$$\mathrm{P}[C \cup D] = \mathrm{P}[C] + \mathrm{P}[D] - \mathrm{P}[C \cap D].$$

By the general multiplication law of probability,

$$\mathrm{P}[C \cap D] = \mathrm{P}[C] \times \mathrm{P}[D \mid C].$$

If one can assume independence of the reliabilities of these components, then

$$\mathrm{P}[C \cup D] = \mathrm{P}[C] + \mathrm{P}[D] - \mathrm{P}[C] \times \mathrm{P}[D].$$

With identical components, $\mathrm{P}[C] = \mathrm{P}[D] = p$. Then

$$\mathrm{P}[C \cup D] = 2p - p^2 = p(2 - p). \tag{1}$$

But the assumption of independence is much more questionable in the case of connection in parallel than it is in the case of connection in series. In the fluid flow context, if one pumping station fails, then that puts more load on the other pumping station, which may increase the likelihood of its failure. The calculation of $\mathrm{P}[C \cup D]$ based on the unconditional probabilities is no longer valid.

Let the (conditional) probability that one component works given that the other component is working be

$$\mathrm{P}[D \mid C] = \mathrm{P}[C \mid D] = a.$$

Knowledge that one component is working may enhance the probability that the other component is working, so that $a > p$.

Let the probability that one component works given that the other component has failed be

$$\mathrm{P}[D \mid \tilde{C}] = \mathrm{P}[C \mid \tilde{D}] = b.$$

Failure of one component may put more strain on the other component, so that $b < p$. In most systems, neither failure nor success will be absolutely certain. The reasonable assumption then follows that $0 < b < p < a < 1$.

Then $P[C \cap D] = P[C] \times P[D \mid C] = P[D] \times P[C \mid D]$

$$\Rightarrow P[C] \times a = P[D] \times a.$$

Equality of the conditional probabilities $P[D \mid C] = P[C \mid D]$ therefore forces equality of the unconditional probabilities $P[D] = P[C] = p$.

$$P[C \cap \tilde{D}] = P[C] \times P[\tilde{D} \mid C] = P[\tilde{D}] \times P[C \mid \tilde{D}]$$

$$\Rightarrow p(1 - a) = (1 - p)b \Rightarrow (1 - a + b)p = b$$

$$\Rightarrow \qquad p = P[C] = P[D] = \frac{b}{1 - (a - b)}. \qquad (2)$$

Note that one has a free choice of only two of $P[D \mid C]$, $P[D \mid \tilde{C}]$ and $P[D]$. Having chosen the value of any two of these probabilities, (2) determines the value of the third probability.

$$P[\tilde{C}] = 1 - p = 1 - \frac{b}{1-(a-b)} = \frac{1-(a-b)-b}{1-(a-b)} = \frac{1-a}{1-(a-b)}.$$

$$P[\tilde{C} \cap \tilde{D}] = P[\tilde{C}] \times P[\tilde{D} \mid \tilde{C}] = (1-p)(1-b)$$

$$= \frac{(1-a)(1-b)}{1-(a-b)} = \frac{1-a-b+ab}{1-a+b}.$$

The probability that the parallel subsystem CD works is therefore $P[C \cup D] = 1 - P[\sim (C \cup D)] = 1 - P[\tilde{C} \cap \tilde{D}]$ (de Morgan's laws)

$$= 1 - \frac{1-a-b+ab}{1-a+b} = \frac{1-a+b-(1-a-b+ab)}{1-(a-b)} = \frac{2b-ab}{1-(a-b)}.$$

Therefore

$$P[C \cup D] = \frac{b(2 - a)}{1 - (a - b)}$$

or

$$P[C \cup D] = \frac{P[D \mid \tilde{C}]\left(2 - P[D \mid C]\right)}{1 - \left(P[D \mid C] - P[D \mid \tilde{C}]\right)}$$

or

$$P[C \cup D] = \frac{P[D \mid \tilde{C}]\left(1 + P[\tilde{D} \mid C]\right)}{1 - \left(P[D \mid \sim C] + P[\sim D \mid C]\right)}. \qquad (3)$$

Figure 3 illustrates the functional relationship between $P[C \cup D]$ and the two conditional probabilities $a = P[D \mid C]$ and $b = P[D \mid \tilde{C}]$, on the domain $0 < b < a < 1$. This plot is available from

http://www.engr.mun.ca/~ggeorge/parallelProb.mw .

When viewed in the appropriate graphical software package, one can view the plot from any desired direction.

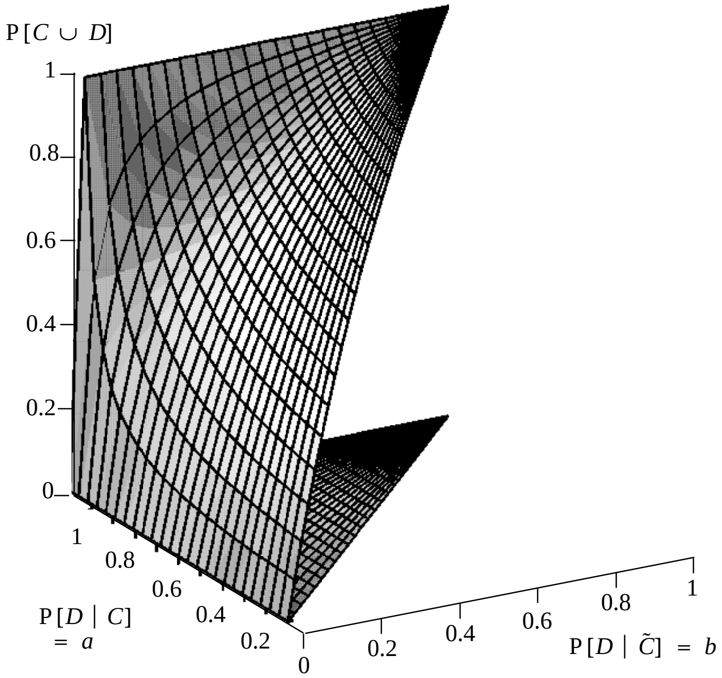

FIGURE 3: Plot of $\mathrm{P}[C \cup D]$ against $\mathrm{P}[D \mid C]$ and $\mathrm{P}[D \mid \tilde{C}]$

In the case of independence, $a = b = p$ and (3) reduces to

$$\mathrm{P}[C \cup D] = \frac{p(2 - p)}{1 - 0}$$

(which is (1)).

One consequence of equations (2) and (3) is the well-known result that the probability of a subsystem in parallel working is greater than the unconditional probability of an individual component working:

$$\mathrm{P}[C \cup D] - \mathrm{P}[C] = \frac{b(2 - a)}{1 - (a - b)} - \frac{b}{1 - (a - b)}$$

$$= \frac{b(2 - a - 1)}{1 - (a - b)} = \frac{b(1 - a)}{1 - (a - b)} > 0$$

for all a, b such that $0 < b < a < 1$.

The general addition and multiplication laws of probability also lead to this result, for any pair of possible events A, B:

$$\mathrm{P}[A \cup B] - \mathrm{P}[A] = \mathrm{P}[B] - \mathrm{P}[A \cap B] = \mathrm{P}[B] - \mathrm{P}[B] \times \mathrm{P}[A \mid B]$$

$$= \mathrm{P}[B]\big(1 - \mathrm{P}[A \mid B]\big) \geqslant 0$$

(with equality only if B is a subset of A).

Example 1

Suppose that each component works 90% of the time when the other component is working, but only 50% of the time when the other component has failed. Then, from equations (2) and (3),

$$a = 0.9,\ b = 0.5 \Rightarrow p = \frac{0.5}{1 - (0.9 - 0.5)} = \frac{5}{6}$$

and

$$\mathrm{P}[C \cup D] = \frac{0.5(2 - 0.9)}{1 - (0.9 - 0.5)} = \frac{0.5 \times 1.1}{1 - 0.4} = \frac{55}{60} = \frac{11}{12}.$$

This subsystem therefore works more than 91% of the time.

Figure 4 illustrates the Venn diagram for this example.

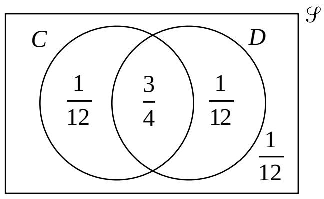

FIGURE 4: Venn diagram for Example 1

One interesting question inspired by this example is:

> Under what circumstances is the probability of the subsystem working greater than the *conditional* probability that one component works given that the other component works?

Put more concisely, when is $\mathrm{P}[C \cup D] > \mathrm{P}[D \mid C]$?

$$\mathrm{P}[C \cup D] - \mathrm{P}[D \mid C] = \frac{b(2 - a)}{1 - (a - b)} - a = \frac{b(2 - a) - a(1 - (a - b))}{1 - (a - b)}.$$

The numerator is

$$2b - ab - a(1 - a) - ab = 2b - 2ab - a(1 - a)$$

$$= 2b(1 - a) - a(1 - a) = (2b - 1)(1 - a).$$

The numerator will be positive only if $2b > a$.

Provided $\mathrm{P}[D \mid C] < 1$,

$$\mathrm{P}[C \cup D] > \mathrm{P}[D \mid C] \text{ if, and only if, } \mathrm{P}[D \mid \tilde{C}] > \tfrac{1}{2}\mathrm{P}[D \mid C] \quad (4)$$

which is the case in Example 1.

Example 2

As an example of the case when $\mathrm{P}[C \cup D] < \mathrm{P}[D \mid C]$, suppose that each component works 80% of the time when the other component is working, but only 30% of the time when the other component has failed.

Then, from (2) and (3),

$$a = 0.8,\ b = 0.3 \Rightarrow p = \frac{0.3}{1 - (0.8 - 0.3)} = \frac{3}{5} = 0.6$$

and

$$\mathrm{P}[C \cup D] = \frac{0.3(2 - 0.8)}{1 - (0.8 - 0.3)} = \frac{0.3 \times 1.2}{0.5} = \frac{36}{50} = \frac{18}{25} = 0.72.$$

This subsystem therefore works 72% of the time, which is less than $\mathrm{P}[D \mid C]$ (but greater than $\mathrm{P}[D]$, as it must be).

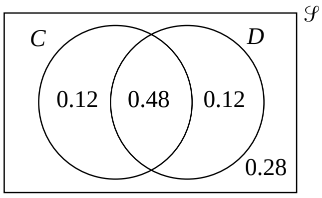

FIGURE 5: Venn diagram for Example 2

Example 3

Suppose that the two conditional probabilities are equally spaced around 0.5. Then $b = 1 - a$ and

$$p = \frac{1 - a}{1 - (a - (1 - a))} = \frac{1}{2}$$

so that success and failure are equally likely for each component in the absence of knowledge of the status of the other component.

$$\mathrm{P}[C \cup D] = \frac{(1 - a)(2 - a)}{1 - (a - (1 - a))} = \frac{2 - a}{2} = \frac{2 - (1 - b)}{2}$$

so that

$$\mathrm{P}[C \cup D] = 1 - \tfrac{1}{2}\mathrm{P}[D \mid C] = \tfrac{1}{2} + \tfrac{1}{2}\mathrm{P}[D \mid \tilde{C}]. \qquad (5)$$

One extreme case occurs when $a = b = p = \frac{1}{2}$, so that the components are independent and $\mathrm{P}[C \cup D] = \frac{3}{4}$ (consistent with (1)).

The other extreme case is when failure of one component guarantees failure of the other and success of one component guarantees success of the other (perfect correlation, $a = 1, b = 0$). Equation (3) is indeterminate, which leads to the gap in the plot at $a = 1, b = 0$ in Figure 3. Using (5) (or the limit for (3)) shows that $\mathrm{P}[C \cup D]$ tends to 0.5 (success and failure equally likely), but events C and $\tilde{D}$ are mutually exclusive (as dependent as possible). In a Venn diagram, C and D are the same set. Examining values close to this extreme case, $a = 0.999$, $b = 0.001$, (2) and (3) lead to $p = \mathrm{P}[C] = \mathrm{P}[D] = 0.5$ and $\mathrm{P}[C \cup D] = 0.5005$.

For all other cases in between these two extremes, $\frac{1}{2} < \mathrm{P}[C \cup D] < \frac{3}{4}$.

Also, from (4), $\mathrm{P}[C \cup D] > \mathrm{P}[D \mid C]$ requires $b > \frac{1}{2}a$.

With the additional constraint that $b = 1 - a$, this condition becomes

$$1 - a > \frac{a}{2} \Rightarrow 1 > \frac{3a}{2} \Rightarrow a < \frac{2}{3}.$$

Therefore, in the case where the two conditional probabilities are equally spaced around 0.5, $\mathrm{P}[C \cup D] > \mathrm{P}[D \mid C]$ requires $\frac{1}{2} \leqslant \mathrm{P}[D \mid C] < \frac{2}{3}$.

Non-identical components in parallel

We can also deduce an expression for $\mathrm{P}[C \cup D]$, in terms of the conditional probabilities only, in the more general case where the two components are *not* identical, so that $\mathrm{P}[D \mid C] \neq \mathrm{P}[C \mid D]$ and $\mathrm{P}[D \mid \tilde{C}] \neq \mathrm{P}[C \mid \tilde{D}]$. However, the algebra is messier! To render the algebra somewhat easier to follow, we use the following abbreviations: $c = \mathrm{P}[C]$, $d = \mathrm{P}[D]$, $r = \mathrm{P}[C \mid D]$, $s = \mathrm{P}[C \mid \tilde{D}]$, $t = \mathrm{P}[D \mid C]$, $u = \mathrm{P}[D \mid \tilde{C}]$.

The total probability partitions into four exhaustive and mutually exclusive probabilities which, by the general multiplication law of probability, are

$$\mathrm{P}[C \cap D] = ct = dr, \qquad \mathrm{P}[\tilde{C} \cap D] = \tilde{c}u = d\tilde{r},$$

$$\mathrm{P}[C \cap \tilde{D}] = c\tilde{t} = \tilde{d}s, \qquad \mathrm{P}[\tilde{C} \cap \bar{D}] = \tilde{c}\tilde{u} = \tilde{d}\tilde{s}, \tag{6}$$

from which $c = c(t + \tilde{t}) = dr + \tilde{d}s$ and $d = d(r + \tilde{r}) = ct + \tilde{c}u$

$$c = dr + \tilde{d}s \Rightarrow dr = ct = (dr + \tilde{d}s)t$$

$$\Rightarrow dr = drt + (1 - d)st \Rightarrow d(r - rt + st) = st \Rightarrow d = \frac{st}{r\tilde{t} + st}. \tag{7}$$

In a similar way, the two unconditional probabilities can then be expressed in terms of the four conditional probabilities and their complements, each in four equivalent ways:

$$\mathrm{P}[C] = c = \frac{ru}{ru + \tilde{r}t} = \frac{s\tilde{u}}{s\tilde{u} + \tilde{s}\tilde{t}} = \frac{rs}{st + r\tilde{t}} = \frac{r\tilde{s}u + \tilde{r}s\tilde{u}}{\tilde{s}u + \tilde{r}\tilde{u}}$$

and

$$\mathrm{P}[D] = d = \frac{st}{st + r\tilde{t}} = \frac{\tilde{s}u}{\tilde{s}u + \tilde{r}\tilde{u}} = \frac{tu}{\tilde{r}t + ru} = \frac{\tilde{s}\tilde{t}u + st\tilde{u}}{\tilde{s}\tilde{t} + s\tilde{u}}. \tag{8}$$

These equations place one constraint on the values of r, s, t, u.

$$\frac{st}{st + r\tilde{t}} = \frac{\tilde{s}u}{\tilde{s}u + \tilde{r}\tilde{u}} \Rightarrow st(\tilde{s}u + \tilde{r}\tilde{u}) = \tilde{s}u(st + r\tilde{t})$$

$$\Rightarrow s\tilde{s}tu + \tilde{r}st\tilde{u} = s\tilde{s}tu + r\tilde{s}\tilde{t}u \Rightarrow \tilde{r}st(1 - u) = r\tilde{s}\tilde{t}u$$

$$\Rightarrow \tilde{r}st = r\tilde{s}\tilde{t}u + \tilde{r}stu \Rightarrow u = \frac{\tilde{r}st}{\tilde{r}st + r\tilde{s}\tilde{t}}. \tag{9}$$

Having chosen values for $r = \mathrm{P}[C \mid D]$, $s = \mathrm{P}[C \mid \tilde{D}]$ and $t = \mathrm{P}[D \mid C]$, there is no freedom of choice for $u = \mathrm{P}[D \mid \tilde{C}]$.

There are several ways to obtain $\mathrm{P}[C \cup D]$, including

$$\mathrm{P}[C \cup D] = \mathrm{P}[C] + \mathrm{P}[\tilde{C} \cap D] = c + d\tilde{r} \text{ (from (6))}$$

and from equation (8),

$$\mathrm{P}[C \cup D] = \frac{rs}{st + r\tilde{t}} + \frac{st}{st + r\tilde{t}} \times \tilde{r} = \frac{s(r + \tilde{r}t)}{st + r\tilde{t}}.$$

Therefore

$$\mathrm{P}[C \cup D] = \frac{\mathrm{P}[C \mid \tilde{D}] \times \left(\mathrm{P}[C \mid D] + \mathrm{P}[\tilde{C} \mid D] \times \mathrm{P}[D \mid C]\right)}{\mathrm{P}[C \mid \tilde{D}] \times \mathrm{P}[D \mid C] + \mathrm{P}[C \mid D] \times \mathrm{P}[\tilde{D} \mid C]}. \quad (10)$$

Example 4

$$\mathrm{P}[C \mid D] = 0.90. \ \mathrm{P}[C \mid \tilde{D}] = 0.75 \text{ and } \mathrm{P}[D \mid C] = 0.50.$$

Equation (9) $\Rightarrow$

$$\mathrm{P}[D \mid \tilde{C}] = \frac{0.10 \times 0.75 \times 0.50}{0.10 \times 0.75 \times 0.50 + 0.90 \times 0.25 \times 0.50} = \frac{0.0375}{0.1500} = 0.25.$$

Equation (8) $\Rightarrow \mathrm{P}[C] = \dfrac{0.90 \times 0.75}{0.75 \times 0.50 + 0.90 \times 0.50} = \dfrac{0.675}{0.825} = \dfrac{9}{11} = 0.\dot{8}\dot{1}$

and $\mathrm{P}[D] = \dfrac{0.75 \times 0.50}{0.75 \times 0.50 + 0.90 \times 0.50} = \dfrac{0.375}{0.825} = \dfrac{5}{11} = 0.\dot{4}\dot{5}.$

Equation (10) $\Rightarrow$

$$\mathrm{P}[C \cup D] = \frac{0.75(0.90 + 0.10 \times 0.50)}{0.75 \times 0.50 + 0.90 \times 0.50} = \frac{0.7125}{0.8250} = \frac{19}{22} = 0.86\dot{3}\dot{6}.$$

While this is greater than $\mathrm{P}[C]$, $\mathrm{P}[D]$ and $\mathrm{P}[D \mid C]$, it is less than $\mathrm{P}[C \mid D]$.

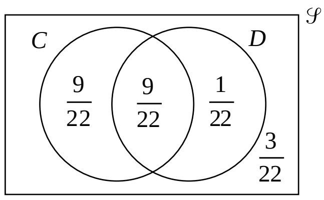

FIGURE 6: Venn diagram for Example 4

When is $\mathrm{P}[C \cup D] > \max\left(\mathrm{P}[D \mid C], \mathrm{P}[C \mid D]\right)$?

$$\mathrm{P}[C \cup D] > \mathrm{P}[D \mid C] \Rightarrow \frac{s(r + \tilde{r}t)}{st + r\tilde{t}} > t$$

$$\Rightarrow s(r + \tilde{r}t) > t(st + r\tilde{t}) \Rightarrow rs + \tilde{r}st > st^2 + rt\tilde{t}$$

$$\Rightarrow s(r + \tilde{r}t - t^2) > rt\tilde{t} \Rightarrow s > \frac{rt\tilde{t}}{r + \tilde{r}t - t^2}.$$

But $r + \tilde{r}t - t^2 = r + t - rt - t^2 = r(1 - t) + t(1 - t) = (r + t)\tilde{t}$

$$\Rightarrow s > \frac{rt}{r + t}$$

and

$$\mathrm{P}[C \cup D] > \mathrm{P}[C \mid D] \Rightarrow \frac{s(r + \tilde{r}t)}{st + r\tilde{t}} > r$$

$$\Rightarrow s(r + \tilde{r}t) > r(st + r\tilde{t}) \Rightarrow rs + \tilde{r}st > rst + rr\tilde{t}$$

$$\Rightarrow s(r + \tilde{r}t - rt) > rr\tilde{t} \Rightarrow s > \frac{r^2\tilde{t}}{r + \tilde{r}t - rt}.$$

But $r + \tilde{r}t - rt = r(1 - t) + \tilde{r}t = r\tilde{t} + \tilde{r}t$

$$\Rightarrow s > \frac{r^2\tilde{t}}{r\tilde{t} + \tilde{r}t}.$$

Therefore $\mathrm{P}[C \cup D] > \max(\mathrm{P}[D \mid C], \mathrm{P}[C \mid D])$ when

$$\mathrm{P}[C \mid \tilde{D}] > \max\left(\frac{\mathrm{P}[C \mid D] \times \mathrm{P}[D \mid C]}{\mathrm{P}[C \mid D] + \mathrm{P}[D \mid C]},\right.$$

$$\left.\frac{(\mathrm{P}[C \mid D])^2 \times \mathrm{P}[\tilde{D} \mid C]}{\mathrm{P}[C \mid D] \times \mathrm{P}[\tilde{D} \mid C] + \mathrm{P}[\tilde{C} \mid D] \times \mathrm{P}[D \mid C]}\right). \tag{11}$$

Obviously this is nowhere near as elegant as the condition (4) in the case of identical components.

Identical components

In the case when the components are identical (interchangeable), $r = t$ and $s = u$. Equation (8) becomes

$$c = \frac{tu}{tu + \tilde{t}t} = \frac{u\tilde{u}}{u\tilde{u} + \tilde{u}\tilde{t}} = \frac{ut}{ut + t\tilde{t}} = \frac{t\tilde{u}u + \tilde{t}u\tilde{u}}{\tilde{u}u + \tilde{t}\tilde{u}}$$

and

$$d = \frac{ut}{ut + t\tilde{t}} = \frac{\tilde{u}u}{\tilde{u}u + \tilde{t}\tilde{u}} = \frac{tu}{\tilde{t}t + tu} = \frac{\tilde{u}\tilde{t}u + ut\tilde{u}}{\tilde{u}\tilde{t} + u\tilde{u}}$$

and all eight expressions in (8) simplify to

$$c = d = \frac{u}{u + \tilde{t}} = \frac{\mathrm{P}[D \mid \tilde{C}]}{\mathrm{P}[D \mid \tilde{C}] + \mathrm{P}[\tilde{D} \mid C]},$$

which is (2).

Equation (10) becomes

$$\mathrm{P}[C \cup D] = \frac{u(t + \tilde{t}t)}{ut + t\tilde{t}} = \frac{ut(1 + \tilde{t})}{t(u + \tilde{t})} = \frac{u(1 + \tilde{t})}{u + \tilde{t}}$$

$$\Rightarrow \mathrm{P}[C \cup D] = \frac{\mathrm{P}[D \mid \tilde{C}] \times (1 + \mathrm{P}[\tilde{D} \mid C])}{\mathrm{P}[D \mid \tilde{C}] + \mathrm{P}[\tilde{D} \mid C]}$$

which recovers (3) above.

Equation (11) reduces to (4):

$$r = t \Rightarrow \frac{rt}{r + t} = \frac{t^2}{t + t} = \frac{t}{2} \text{ and } \frac{r^2\tilde{t}}{r\tilde{t} + \tilde{r}t} = \frac{t^2\tilde{t}}{t\tilde{t} + \tilde{t}t} = \frac{t}{2}.$$

An Excel spreadsheet to calculate $\text{P}[C \cup D]$ directly from the three conditional probabilities $\text{P}[C \mid D]$, $\text{P}[C \mid \tilde{D}]$ and $\text{P}[D \mid C]$ can be downloaded from

http://www.engr.mun.ca/~ggeorge/Parallels.xlsx

Conclusion

The main contributions here are (10) (and its special case (3)), relating $\text{P}[C \cup D]$ directly to the conditional probabilities $\text{P}[C \mid D]$, $\text{P}[C \mid \tilde{D}]$ and $\text{P}[D \mid C]$ only, and (4), providing the condition, in the case of identical components, for $\text{P}[C \cup D]$ to exceed $\text{P}[D \mid C]$.

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GLYN GEORGE

Department of Electrical and Computer Engineering, Memorial University of Newfoundland, St. John's NL, Canada A1B 3X5

e-mail: *glyn@mun.ca*